Chapter 5

Topology

In this chapter, all linear spaces and flat spaces under consideration are assumed to be finite-dimensional except when a statement to the contrary is made.

51 Cells and Norms

We assume that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given. Given any two distinct points $x, y \in \mathcal{E}$, we define the **open segment** joining x and y to be the set of all flat combinations of (x, y) with strictly positive coefficients and denote it by

$$]\, x, y\, [\; := \left\{ \lambda x + \mu y \mid \lambda, \mu \in \mathbb{P}^{\times}, \lambda + \mu = 1 \right\}. \tag{51.1}$$

If $s, t \in \mathbb{R}$ and $s < t$, the notation (51.1) is consistent with the notation $]\, x, t\, [\; := \{r \in \mathbb{R} \mid s < r < t\}$ (see (08.16)). If $[x, y]$ is the segment joining x and y according to the definition (37.1), then

$$]\, x, y\, [\; = [x, y] \setminus \{x, y\}. \tag{51.2}$$

The point $\frac{1}{2}x + \frac{1}{2}y \in \;]\, x, y\, [$ is called the *midpoint* not only of the pair (x, y), but also of the segments $]\, x, y\, [$ and $[x, y]$.

Definition 1: *A non-empty subset $\mathcal{C}$ of $\mathcal{E}$ is called a* **cell** *centered at $q \in \mathcal{E}$ if $\mathcal{C}$ is convex and if every line through q intersects $\mathcal{C}$ in an open segment whose midpoint is q. A cell in $\mathcal{V}$ centered at $\mathbf{0} \in \mathcal{V}$ is called a* **norming cell** *of $\mathcal{V}$.*

The following facts are immediate consequences of the definition.

Proposition 1: *Let $\mathcal{B}$ be a norming cell; then $-\mathcal{B} = \mathcal{B}$ and $\lambda\mathcal{B} + \mu\mathcal{B} = (\lambda + \mu)\mathcal{B}$ for all $\lambda, \mu \in \mathbb{P}$, also, if $t \in \mathbb{R}^\times$, then $t\mathcal{B}$ is again a norming cell.*

Proposition 2: *The intersection of a non-empty finite collection of norming cells is again a norming cell.*

Defition 2: *The* **closure** *of a norming cell $\mathcal{B}$ is defined to be*

$$\overline{\mathcal{B}} := \bigcap \left\{ t\mathcal{B} \mid t \in 1 + \mathbb{P}^\times \right\}. \tag{51.3}$$

It is clear that $\overline{\mathcal{B}}$ includes $\mathcal{B}$ and, being the intersection of a collection of convex setrs, is convex (see Prop. 1 of Sect. 3.7).

The following results are immediate from the definitions.

Proposition 3: *A non-empty subset $\mathcal{B}$ of $\mathcal{V}$ is a norming cell if and only if it is convex and for all $\mathbf{u} \in \mathcal{V}^\times$ there is an $r \in \mathbb{P}^\times$ such that*

$$\mathcal{B} \cap \mathbb{R}\mathbf{u} = (\,]-r, r\,[\,)\,\mathbf{u} = \,]-r\mathbf{u}, r\mathbf{u}\,[\,. \tag{51.4}$$

If $\mathcal{B}$ is a norming cell and if (51.4) *holds then*

$$\overline{\mathcal{B}} \cap \mathbb{R}\mathbf{u} = ([-r, r])\,\mathbf{u} = [r\mathbf{u}, r\mathbf{u}]\,. \tag{51.5}$$

Proposition 4: *If $\mathcal{B}$ is a norming cell and $r \in [0, 1\,[$ then*

$$\mathcal{B} = \bigcup_{t \in [r,1\,[} t\mathcal{B} = \bigcup_{t \in [r,1\,[} t\overline{\mathcal{B}} \tag{51.6}$$

and

$$\overline{\mathcal{B}} := \bigcap_{t \in 1+\mathbb{P}^\times} t\mathcal{B} = \bigcap_{t \in 1+\mathbb{P}^\times} t\overline{\mathcal{B}}. \tag{51.7}$$

Definition 3: *A function $\nu : \mathcal{V} \to \mathbb{P}$ is called a* **norm** *on $\mathcal{V}$ if*

(N_1) $\nu(s\mathbf{v}) = |s|\,\nu(\mathbf{v})$ for all $s \in \mathbb{R},\ \mathbf{v} \in \mathcal{V}$,
(N_2) $\nu(\mathbf{u} + \mathbf{v}) \leq \nu(\mathbf{u}) + \nu(\mathbf{v})$ for all $\mathbf{u,v} \in \mathcal{V}$,
(N_3) $\nu(\mathbf{v}) = 0 \Longrightarrow \mathbf{v} = \mathbf{0}$.

The term "norming cell" is justified by the following result.

Proposition 5: *If ν is a norm on $\mathcal{V}$, then*

$$\mathrm{Ce}(\nu) := \nu^<([0, 1\,[\,) = \{\mathbf{v} \in \mathcal{V} \mid \nu(\mathbf{v}) < 1\} \tag{51.8}$$

is a norming cell. Its closure is

$$\overline{\mathrm{Ce}}(\nu) := \overline{\mathrm{Ce}(\nu)} = \nu^{<}([0,1]) = \{\mathbf{v} \in \mathcal{V} \mid \nu(\mathbf{v}) \leq 1\}, \tag{51.9}$$

and for each $\mathbf{u} \in \mathcal{V}^{\times}$ *we have*

$$\{s \in \mathbb{R} \mid s\mathbf{u} \in \mathrm{Ce}(\nu)\} = \left]-\frac{1}{\nu(\mathbf{u})}, \frac{1}{\nu(\mathbf{u})}\right[\tag{51.10}$$

and

$$\{s \in \mathbb{R} \mid s\mathbf{u} \in \overline{\mathrm{Ce}}(\nu)\} = \left[-\frac{1}{\nu(\mathbf{u})}, \frac{1}{\nu(\mathbf{u})}\right]. \tag{51.11}$$

Conversely, if $\mathcal{B}$ *is a norming cell in* $\mathcal{V}$*, then* $\{t \in \mathbb{P} \mid \mathbf{v} \in t\mathbf{B}\} \neq \emptyset$ *for every* $\mathbf{v} \in \mathcal{V}$ and $\mathrm{no}_{\mathcal{B}} : \mathcal{V} \to \mathbb{P}$, *defined by*

$$\mathrm{no}_{\mathcal{B}}(\mathbf{v}) := \inf\{t \in \mathbb{P} \mid \mathbf{v} \in t\mathcal{B}\}, \tag{51.12}$$

is a norm on $\mathcal{V}$.

We have $\mathrm{no}_{\mathrm{Ce}_\nu} = \nu$ *and* $\mathrm{Ce}(\mathrm{no}_{\mathcal{B}}) = \mathcal{B}$.

Proof: Assume that ν is a norm on $\mathcal{V}$. Let $\mathbf{u}$,$\mathbf{v} \in \mathrm{Ce}(\nu)$ be given, so that $\nu(\mathbf{u}) < 1$, $\nu(\mathbf{v}) < 1$. Given $\mathbf{w} \in [\mathbf{u}, \mathbf{v}]$, we have $\mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v}$ for some $\lambda, \mu \in \mathbb{P}$ such that $\lambda + \mu = 1$ (see (37.1)). Hence, by (N_1) and (N_2), we obtain

$$\nu(\mathbf{w}) \leq \nu(\lambda\mathbf{u}) + \nu(\mu\mathbf{v}) = \lambda\nu(\mathbf{u}) + \lambda\nu(\mathbf{v}) < \lambda + \mu = 1,$$

which shows that $\mathbf{w} \in \mathrm{Ce}(\nu)$. Since $\mathbf{u}$,$\mathbf{v} \in \mathrm{Ce}(\nu)$ and $\mathbf{w} \in [\mathbf{u}, \mathbf{v}]$ were arbitrary, it follows that $Ce(\nu)$ is convex. Now let $\mathbf{u} \in \mathcal{V}^{\times}$ be given. Then $\nu(\mathbf{u}) \neq 0$ by (N_3) and hence, by (N_1), we have

$$\begin{aligned} \mathrm{Ce}(\nu) \cap \mathbb{R}\mathbf{u} &= \{s\mathbf{u} \mid s \in \mathbb{R}, \nu(s\mathbf{u}) < 1\} = \{s\mathbf{u} \mid s \in \mathbb{R}, |s|\,\nu(\mathbf{u}) < 1\} \\ &= \left\{s\mathbf{u} \mid s \in \mathbb{R}, |s| < \tfrac{1}{\nu(\mathbf{u})}\right\} = \left]-\tfrac{1}{\nu(\mathbf{u})}, \tfrac{1}{\nu(\mathbf{u})}\right[\,\mathbf{u}. \end{aligned}$$

By Prop. 3, it follows that $\mathrm{Ce}(\nu)$ is a norming cell and that (51.10) holds.

Let $\mathbf{v} \in \mathcal{V}$ be given. By the definition (51.3) we have $\mathbf{v} \in \overline{\mathrm{Ce}}(\nu)$ if and only if $\mathbf{v} \in t\mathrm{Ce}(\nu)$ for all $t \in 1 + \mathbb{P}^{\times}$. Now, given $t \in 1 + \mathbb{P}^{\times}$, we have

$$\begin{aligned} \mathbf{v} \in t\mathrm{Ce}(\nu) &\iff \tfrac{1}{t}\mathbf{v} \in \mathrm{Ce}(\nu) \iff \nu\left(\tfrac{1}{t}\mathbf{v}\right) < 1 \\ &\iff \tfrac{1}{t}\nu(\mathbf{v}) < 1 \iff \nu(\mathbf{v}) < t. \end{aligned}$$

Since $\nu(\mathbf{v}) < t$ holds for all $t \in 1+\mathbb{P}^{\times}$ if and only if $\nu(\mathrm{v}) \leq 1$, it follows that $\mathbf{v} \in \overline{\mathrm{Ce}}(\nu)$ if and only if $\nu(\mathbf{v}) \leq 1$, which proves (51.9).

Let $\mathbf{u} \in \mathcal{V}^{\times}$ be given. By (N_3), l we have $\nu(\mathbf{u}) \neq 0$ and hence, by (51.9),

$$\begin{aligned}\overline{\mathrm{Ce}}(\nu) \cap \mathbb{R}\mathbf{u} &= \{s\mathbf{u} \mid s \in \mathbb{R},\ \nu(s\mathbf{u}) \leq 1\} \\ &= \{s\mathbf{u} \mid s \in \mathbb{R}, |s|\nu(\mathbf{u}) \leq 1\} \\ &= \left\{s\mathbf{u} \mid s \in \mathbb{R}, |s| \leq \tfrac{1}{\nu(\mathbf{u})}\right\} \\ &= \left[-\tfrac{1}{\nu(\mathbf{u})}, \tfrac{1}{\nu(\mathbf{u})}\right]\mathbf{u},\end{aligned}$$

which proves (51.11).

Assume that $\mathcal{B}$ is a norming cell. Let $\mathbf{v} \in \mathcal{V}$ be given. If $\mathbf{v} = 0$, then $\mathbf{v} \in t\mathcal{B}$ for all $t \in \mathbb{P}$. Hence $\{t \in \mathbb{P} \mid \mathbf{v} \in t\mathcal{B}\} = \mathbb{P}$, and (51.12) gives $\mathrm{no}_{\mathcal{B}}(\mathbf{v}) = 0$. Assume, then, that $\mathbf{v} \neq 0$ and let $r \in \mathbb{P}^{\times}$ be determined according to Prop. 3. By (51.4) we then have, for all $s \in \mathbb{P}^{\times}$, $s\mathbf{v} \in \mathcal{B} \Leftrightarrow s < r$ and hence, for all $t \in \mathbb{P}^{\times}$, $\mathbf{v} \in t\mathcal{B} \Leftrightarrow \frac{1}{t} < r \Leftrightarrow t > \frac{1}{r}$. Therefore we have $\{t \in \mathbb{P} \mid \mathbf{v} \in t\mathcal{B}\} = \frac{1}{r} + \mathbb{P}^{\times} \neq \emptyset$, and (51.12) gives $\mathrm{no}_{\mathcal{B}}(\mathbf{v}) = \frac{1}{r} \neq 0$. We conclude that $\nu := \mathrm{no}_{\mathcal{B}}$ is meaningful and satisfies (N_3).

Let $s \in \mathbb{R}^{\times}$ and $\mathbf{v} \in \mathcal{V}$ be given. Since $\mathcal{B} = -\mathcal{B}$ by Prop. 1, we have $s\mathcal{B} = |s|\mathcal{B}$ and hence, for all $t \in \mathbb{P}$,

$$\mathbf{v} \in t\mathcal{B} \Longleftrightarrow s\mathbf{v} \in |s|t\mathcal{B}.$$

It follows, by (51.12), that

$$\nu(s\mathbf{v}) = \inf\{|s|t \mid \mathbf{v} \in t\mathcal{B}\} = |s|\nu(\mathbf{v})$$

and hence that ν satisfies (N_1).

Now let $\mathbf{u},\mathbf{v} \in \mathcal{V}$ be given. Given $\lambda \in \nu(\mathbf{u}) + \mathbb{P}^{\times}$ and $\mu \in \nu(\mathbf{v}) + \mathbb{P}^{\times}$, we have, by (51.12), $\mathbf{u} \in \lambda\mathcal{B}$ and $\mathbf{v} \in \mu\mathcal{B}$. Using Prop. 1, it follows that $\mathbf{u} + \mathbf{v} \in \lambda\mathcal{B} + \mu\mathcal{B} = (\lambda + \mu)\mathcal{B}$ and hence, by (51.12) again, that $\nu(\mathbf{u} + \mathbf{v}) \leq \lambda + \mu$. Since $\lambda \in \nu(\mathbf{u}) + \mathbb{P}^{\times}$ and $\mu \in \nu(\mathbf{v}) + \mathbb{P}^{\times}$ were arbitrary, it follows that $\nu(\mathbf{u} + \mathbf{v}) \leq \nu(\mathbf{u}) + \nu(\mathbf{v})$, i.e., that ν satisfies (N_2). ■

The "subaddivity law" (N_2) extends, of course, to sums of arbitrary finite families (see Sect. 07): If ν is a norm on $\mathcal{V}$ and $(\mathbf{u}_i \mid i \in I)$ a finite family in $\mathcal{V}$, then

$$\nu\left(\sum_{i\in I}\mathbf{u}_i\right) \leq \sum_{i\in I}\nu\left(\mathbf{u}_i\right). \tag{51.13}$$

The following results are immediate from Def. 3 and Prop. 5.

Proposition 6: *If ν is a norm on $\mathcal{V}$ and $r \in \mathbb{P}^{\times}$, then $r\nu$ (defined by $(r\nu)(\mathbf{v}) = r(\nu(\mathbf{v}))$ for all $\mathbf{v} \in \mathcal{V}$) is again a norm on $\mathcal{V}$ and* $\mathrm{Ce}(r\nu) = \frac{1}{r}\mathrm{Ce}(\nu)$. *If $\mathcal{B}$ is a norming cell in $\mathcal{V}$ and $t \in \mathbb{R}^{\times}$, then* $\mathrm{no}_{(t\mathcal{B})} = \frac{1}{|t|}\mathrm{no}_{\mathcal{B}}$.

It is clear that a subset $\mathcal{C}$ of $\mathcal{E}$ is a cell centered at $q \in \mathcal{E}$ if and only if $\mathcal{C} - q$ is a norming cell. If $\mathcal{B}$ is a norming cell, $q \in \mathcal{E}$, and $\sigma \in \mathbb{P}^{\times}$, then $q + \sigma\mathcal{B}$ is a cell centered at q.

Definition 4: *If $\mathcal{B}$ is a norming cell of $\mathcal{V}$, then every cell of the form $q + \sigma\mathcal{B}$, with $a \in \mathcal{E}$, $\sigma \in \mathbb{P}^{\times}$, is called a* **cell modelled** *on $\mathcal{B}$ of* **scale** *σ. The* **closure** *of a cell of this form is defined to be $q + \sigma\overline{\mathcal{B}}$, where $\overline{\mathcal{B}}$ is defined by Def. 2.*

If ν is a norm on $\mathcal{V}$, $q \in \mathcal{E}$, and $\sigma \in \mathbb{P}^{\times}$, then the cell

$$q + \sigma\mathrm{Ce}(\nu) = \{x \in \mathcal{E} \mid \nu(x - q) < \sigma\}, \tag{51.14}$$

modelled on $\mathrm{Ce}(\nu)$, *is called the ν-***cell** *of scale σ centered at q.*

In view of (51.9), the closure of the cell (51.14) is given by

$$q + \sigma\overline{\mathrm{Ce}}(\nu) = \{x \in \mathcal{E} \mid \nu(x - q) \leq \sigma\}. \tag{51.15}$$

Remark: If $\dim \mathcal{E} = \dim \mathcal{V} = 0$ then $\mathcal{E}$ and $\mathcal{V}$ are singletons, the only cell in $\mathcal{E}$ is $\mathcal{E}$ itself and the only norming cell is $\mathcal{V}$. The corresponding norm is the constant 0. The conditions of Def. 1 are satisfied in this case because $\mathcal{E}$ and $\mathcal{V}$ include no lines at all. The express requirement that a cell be non-empty is needed only for this case. If $\dim \mathcal{E} = \dim \mathcal{V} > 0$, then $\mathcal{E}$ and $\mathcal{V}$ are never cells.

In a few of the formulas below we must tacitly assume that $\dim \mathcal{E} = \dim \mathcal{V} > 0$, becase the empty family is the only basis of $\mathcal{V}$ where $\dim \mathcal{V} = 0$. All the results of this chapter become trivial when $\dim \mathcal{E} = \dim \mathcal{V} = 0$. In some formulas, such as (52.1) and (52.4), one must remember that the supremum of the empty set, when regarded as a subset of $\mathbb{P}$ or $\overline{\mathbb{P}}$, is 0. ■

Examples:

(A) Boxes. Let $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ be a basis of $\mathcal{V}$ and let $\mathbf{b}^* := (\mathbf{b}_i^* \mid i \in I)$ be its dual basis. The **norming box** determinged by $\mathbf{b}$ is defined to be

$$\begin{aligned}\mathrm{Box}(\mathbf{b}) &= \bigcap_{i\in I}(\mathbf{b}_i^*)^{<}(\,]-1,1[\,)\\ &= \left\{\mathbf{v}\in\mathcal{V}\,\middle|\,|\mathbf{b}_i^*\mathbf{v}|<1 \text{ for all } i\in I\right\}\\ &= (\mathrm{lnc}_{\mathbf{b}})_{>}\left(\,]-1,1[^{I}\right)\\ &= \left\{\sum_{i\in I}\lambda_i\mathbf{b}_i\,\middle|\,\lambda\in\mathbb{R}^I, |\lambda_i|<1 \text{ for all } i\in I\right\}.\end{aligned}$$

(The case when $\dim\mathcal{V}=2,\ I=\{1,2\}$ is illustrated in Figure 1.) The closure of Box($\mathbf{b}$) is

$$\overline{\mathrm{Box}(\mathbf{b})} = \left\{\sum_{i\in I}\lambda_i\mathbf{b}_i\,\middle|\,\lambda\in\mathbb{R}^I, |\lambda_i|\le 1 \text{ for all } i\in I\right\}$$

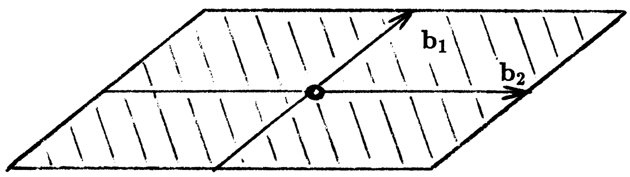

Figure 1.

The norm corresponding to Box($\mathbf{b}$) is given by

$$\mathrm{no}_{\mathrm{Box}(\mathbf{b})}(\mathbf{v}) = \max\{|\mathbf{b}_i^*\mathbf{v}| \mid i\in I\} \tag{51.16}$$

A **box** in $\mathcal{E}$ is defined to be a cell modelled on a norming box, i.e., a set of the form

$$q+\sigma\mathrm{Box}(\mathbf{b}) = \left\{x\in\mathcal{E}\,\middle|\,|\mathbf{b}_i^*(x-q)|<\sigma \text{ for all } i\in I\right\} \tag{51.17}$$

with $q=\mathcal{E},\ \sigma\in\mathbb{P}^{\times}$.

(B) Diamonds. Let $\mathbf{b}$ be a basis of $\mathcal{V}$, with dual $\mathbf{b}^*$, as in example (A). The **norming diamond** determined by $\mathbf{b}$ is defined to be

$$\begin{aligned}
\operatorname{Dmd}(\mathbf{b}) &:= \left(\sum_{i\in I}|\mathbf{b}_i^*|\right)^{<}(]-1,1[) \\
&= \left\{\mathbf{v}\in\mathcal{V}\,\middle|\,\sum_{i\in I}|\mathbf{b}_i^*\mathbf{v}|<1\right\} = \left\{\sum_{i\in I}\lambda_i\mathbf{b}_i\,\middle|\,\lambda\in\mathbb{R}^I,\ \sum_{i\in I}|\lambda_i|<1\right\}.
\end{aligned}$$

(The case when $\dim\mathcal{V}=2,\ I=\{1,2\}$ is illustrated in Figure 2.) The closure of $\operatorname{Dmd}(\mathbf{b})$ is

$$\overline{\operatorname{Dmd}}(\mathbf{b}) = \left\{\sum_{i\in I}\lambda_i\mathbf{b}_i\,\middle|\,\lambda\in\mathbb{R}^I,\ \sum_{i\in I}|\lambda_i|\le 1\right\}.$$

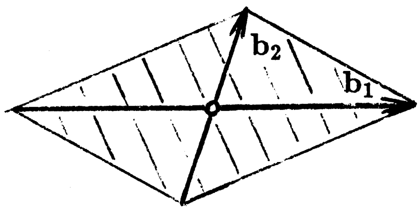

Figure 2.

The norm corresponding to $\operatorname{Dmd}(\mathbf{b})$ is given by

$$\operatorname{no}_{\operatorname{Dmd}(\mathbf{b})}(\mathbf{v}) = \sum_{i\in I}|\mathbf{b}_i^*\mathbf{v}|\,. \tag{51.18}$$

A **Diamond** in $\mathcal{E}$ is defined to be a cell modelled on a norming diamond, i.e., a set of the form

$$q+\rho\operatorname{Dmd}(\mathbf{b}) = \left\{x\in\mathcal{E}\,\middle|\,\sum_{i\in I}|\mathbf{b}_i^*(x-q)|<\rho\right\}. \tag{51.19}$$

(C) Balls. Assume that $\mathcal{E}$ is a genuine Euclidean space, so that $\mathcal{V}$ has the structure of a genuine inner product space. Then the magnitude-function $|\cdot|:\mathcal{V}\to\mathbb{P}$ (defined by $|\mathbf{v}|:=\sqrt{\mathbf{v}\cdot\mathbf{v}}$ for all $\mathbf{v}\in\mathcal{V}$) is a norm. The corresponding norming cell is the unit ball $\operatorname{Ubl}\mathcal{V}$ defined by (42.7). Its closure in the sense of Def. 2 is the closed unit ball $\overline{\operatorname{Ubl}}\mathcal{V}$ defined by (42.8). A cell modelled on $\operatorname{Ubl}\mathcal{V}$ the sense of Def. 4 is just a ball in $\mathcal{E}$ as defined by (46.6) and its closure is the closed ball defined by (46.7). Also, a $|\cdot|$-cell of scale ρ centered at q in the sense of Def. 4 is just a ball of radius ρ centered at q. ■

The proof of the following result is based on the Halfspace-Inclusion Theorem of Sect. 38.

Proposition 7: *Let $\mathcal{B}$ be a norming cell and $\mathcal{U}$ a one-dimensional subspace of $\mathcal{V}$. Then there is a linear form $\boldsymbol{\lambda} \in \mathcal{V}^*$ such that*

$$\boldsymbol{\lambda}_>(\mathcal{B}) = \boldsymbol{\lambda}_>(\mathcal{B} \cap \mathcal{U}) = \,]-1, 1[\,. \tag{51.20}$$

Proof: By the definition of a cell, there is a $\mathbf{u} \in \mathcal{U}^\times$ such that $\mathcal{B} \cap \mathcal{U} = \,]-\mathbf{u}, \mathbf{u}[$ and hence $(\mathcal{B} + \mathbf{u}) \cap \mathcal{U} = \,]\mathbf{0}, 2\mathbf{u}[$. Since $\mathcal{B} + \mathbf{u}$ is convex and $\mathbf{0} \notin \mathcal{B} + \mathbf{u}$, we can apply the Halfspace-Inclusion Theorem to obtain a $\boldsymbol{\gamma} \in \mathcal{V}^\times$ such that $\boldsymbol{\gamma}_>(\mathcal{B}+\mathbf{u}) \subset \mathbb{P}$. Since $\boldsymbol{\gamma} \neq \mathbf{0}$, there is a $\mathbf{v} \in (\mathcal{V}^*)^\times$, such that $\boldsymbol{\gamma}\mathbf{v} \in \mathbb{P}^\times$. In view of Prop. 3 there is a $t \in \mathbb{P}^\times$ such that $-t\mathbf{v} \in \mathcal{B}$ and hence $-t\mathbf{v} + \mathbf{u} \in \mathcal{B} + \mathbf{u}$. Thus we have $0 \leq \boldsymbol{\gamma}(-t\mathbf{v} + \mathbf{u}) = -t(\boldsymbol{\gamma}\mathbf{v}) + \boldsymbol{\gamma}\mathbf{u}$, which implies $\boldsymbol{\gamma}\mathbf{u} > 0$. It is easily verified, using (51.4), that $\boldsymbol{\lambda} := \frac{1}{(\boldsymbol{\gamma}\mathbf{u})}\boldsymbol{\gamma}$ has the desired property (51.20). ■

Proposition 8: *For every norming cell $\mathcal{B}$ in $\mathcal{V}$ and every $\boldsymbol{\lambda} \in \mathcal{V}^*$, the image $\boldsymbol{\lambda}_>(\mathcal{B})$ of $\mathcal{B}$ under $\boldsymbol{\lambda}$ is a bounded subset of $\mathbb{R}$.*

Proof: Put

$$\mathcal{A} := \left\{ \boldsymbol{\lambda} \in \mathcal{V}^* \,\middle|\, \boldsymbol{\lambda}_>(\mathcal{B}) \text{ is bounded} \right\}.$$

Let $\mathbf{u} \in \mathcal{V}^\times$ be given. Applying Prop. 7 to the case when $\mathcal{U} := \mathbb{R}\mathbf{u}$, we see that there is a $\boldsymbol{\lambda} \in \mathcal{A}$ such that $\boldsymbol{\lambda}\mathbf{u} \neq 0$. Since $\mathbf{u} \in \mathcal{V}^\times$ was arbitrary, we conclude that $\mathcal{A}^\perp = \{0\}$. Hence, by (22.5), we conclude that $\mathcal{A} = (\mathcal{A}^\perp)^\perp = \{0\}^\perp = \mathcal{V}^*$. ■

Norm-Equivalaence Theorem: *for all norms ν and ν' on $\mathcal{V}$ there are $h, k \in \mathbb{P}^\times$ it such that $h\nu \leq \nu' \leq k\nu$ (value-wise).*

Proof: We choose a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ of $\mathcal{V}$, with dual $\mathbf{b}^*{:=}(\mathbf{b}_i^* \mid i \in I)$.

Let $\mathbf{v} \in \mathcal{V}$ be given. We then have $\mathbf{v} = \sum(\lambda_i \mathbf{b}_i \mid i \in I)$ with $\lambda := (\mathbf{b}_i^*\mathbf{v} \mid i \in I) \in \mathbb{R}^I$. Using the properties of (N_1) and (51.13) for the norm ν', we obtain

$$\nu'(\mathbf{v}) \leq \sum_{i \in I} |\lambda_i| \nu'(\mathbf{b}_i) \leq \left(\max_{i \in I} |\lambda_i| \right) \sum_{i \in I} \nu'(\mathbf{b}_i).$$

In view of (51.16), and since $\mathbf{v} \in \mathcal{V}$ was arbitrary, we obtain

$$\nu' \leq s\ \mathrm{no}_{\mathrm{Box}(\mathbf{b})}, \tag{51.21}$$

where $s := \sum(\nu'(\mathbf{b}_i) \mid i \in I) \in \mathbb{P}^\times$.

On the other hand, by Prop. 8, $(\mathbf{b}_i^*)_>(\mathrm{Ce}(\nu))$ is a bounded non-empty subset of $\mathbb{R}$ for each $i \in I$ and hence

$$t := \max_{i \in I} \sup\{|\mathbf{b}_i^* \mathbf{u}| \,\Big|\, \mathbf{u} \in \mathrm{Ce}(\nu)\}.$$

Now let $\mathbf{v} \in \mathcal{V}^\times$ be given. If we put $\mathbf{u} := \frac{\mathbf{v}}{2\nu(\mathbf{v})}$, then $\nu(\mathbf{u}) = \frac{1}{2} < 1$ and hence $\mathbf{u} \in \mathrm{Ce}(\nu)$. It follows that

$$\mathrm{no}_{\mathrm{Box}(\mathbf{b})} \leq 2t\nu. \tag{51.22}$$

Combining (51.21) and (51.22), we find that $\nu' \leq k\nu$ with $k := 2st \in \mathbb{P}^\times$. Interchanging the roles of ν and ν' and replacing $\frac{1}{k}$ by h, we see that also $h\nu \leq \nu'$ for some $h \in \mathbb{P}^\times$. ∎

Using Prop. 5, we immediately obtain the following corollaries.

Corollary 1: *If $\mathcal{B}$ and $\mathcal{B}'$ are norming cells in $\mathcal{V}$, there are $\rho, \sigma \in \mathbb{P}^\times$ such that*

$$\rho\mathcal{B} \subset \mathcal{B}' \subset \sigma\mathcal{B}.$$

Corollary 2: *Let ν and ν' be norms on $\mathcal{V}$ and q be a point in $\mathcal{E}$. Then every ν-cell centered at q includes a ν'-cell centered at q and is included in a ν-cell centered at q.*

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be flat spaces with translation spaces $\mathcal{V}_1$ and $\mathcal{V}_2$. Recall that $\mathcal{E}_1 \times \mathcal{E}_2$ has the structure of a flat space with translation space $\mathcal{V}_1 \times \mathcal{V}_2$ (see end of Sect. 32). It is easily seen that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are norming cells in $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively, then $\mathcal{B}_1 \times \mathcal{B}_2$ is a norming cell in $\mathcal{V}_1 \times \mathcal{V}_2$. If $\nu_1 := \mathrm{no}_{\mathcal{B}_1}, \nu_2 := \mathrm{no}_{\mathcal{B}_2}$ and $\nu := \mathrm{no}_{\mathcal{B}_1 \times \mathcal{B}_2}$, then

$$\nu((\mathbf{v}_1, \mathbf{v}_2)) = \max\{\nu_1(\mathbf{v}_1), \nu_2(\mathbf{v}_2)\} \quad \text{for all} \quad (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{B}_1 \times \mathcal{B}_2. \tag{51.23}$$

Let $\mathcal{F}$ be a flat in $\mathcal{E}$. Since $\mathcal{F}$ has the structure of a flat space, it makes sense to speak of cells in $\mathcal{F}$. It is immediate that if $\mathcal{C}$ is a cell in $\mathcal{E}$ centered at a point $q \in \mathcal{F}$, then $\mathcal{C} \cap \mathcal{F}$ is a cell in $\mathcal{F}$ centered at q. Conversely, if $\mathcal{C}'$ is a cell in $\mathcal{F}$ centered at $q \in \mathcal{F}$, we can construct a cell $\mathcal{C}$ in $\mathcal{E}$ such that $\mathcal{C}' = \mathcal{C} \cap \mathcal{F}$ as follows: We choose a supplement $\mathcal{W}$ of the direction space of $\mathcal{F}$ and a norming cell $\mathcal{B}$ of $\mathcal{W}$. Then $\mathcal{C} := \mathcal{C}' + \mathcal{B}$ is easily seen to be a cell in $\mathcal{E}$ such that $\mathcal{C}' = \mathcal{C} \cap \mathcal{F}$.

Notes 51

(1) The term "ball" is often used for our "cell" and "unit ball" for our "norming cell". I prefer to limit the use of "ball" and "unit ball" to the special case discussed in Example (C).

(2) The notation $\|\mathbf{u}\|$ is commonly used for the value of a norm at $\mathbf{u}$. This notation makes it hard to deal with all possible norms, as we do here. I like to reserve the use of double bars for operator-norms as defined in Sect. 5.2. See also Note 2 to Sect. 42.

(3) What we call a "box" is often called a "parallelotope", a "parallelepiped" (if the dimension is 3), or a "parallelogram" (if the dimension is 2).

(4) What we call a "diamond" is sometimes called a "cross-polytope".

(5) When $\mathcal{V} := \mathbb{R}^I$ for some finite set I and when $\mathbf{b} := \delta^I$, the standard basis defined by (16.2), then the box-norm and diamond-norm are often called "l^∞-norm" and "l^1-norm", respectively. However, these latter terms are more often used in an analogous but infinite-dimensional situation.

(6) No treatment of the topology of finite-dimensional spaces that I am aware of deals with the Norm-Equivalence Theorem in the way I do here. In the conventional treatments a fixed norm (often the magnitude norm on $\mathbb{R}^n$) is assumed to be prescribed, and the definitions of the topological concepts all depend, at first view, on that norm. On occasion the Norm-Equivalence Theorem is proved as an afterthought, and the proof is then based on theorems involving compactness. The definitions of the topological concepts I gave in this chapter do not depend, even at first view, on the choice of any particular norm. I need the Norm-Equivalence Theorem long before I can define compactness. My proof is based on the Half-Space Inclusion Theorem, which does not involve topology at all.

52 Bounded Sets, Operator Norms

Definition 1: *A subset $\mathcal{S}$ of a flat space $\mathcal{E}$ is said to be* **bounded** *if for every $a \in Flf(\mathcal{E})$, the image $a_>(\mathcal{S})$ is a bounded subset of* $\mathbb{R}$ (see Sect. 36).

If $\mathbb{R}$ is regarded as a flat space, this definition is clearly consistent with the usual definition of boundedness of subsets of $\mathbb{R}$ (see Sect. 08).

It is evident that every subset of a bounded set is bounded, that finite sets are bounded, and that the union of a finite collection of bounded sets is again bounded.

The following result is immediate from the definition.

Proposition 1: *The image of a bounded set under a flat mapping is again bounded. If $\mathcal{E}$ is a flat space with translation space $\mathcal{V}$ and if $\mathbf{v} \in \mathcal{V}$ and $x \in \mathcal{E}$, then a subset $\mathcal{S}$ of $\mathcal{E}$ is bounded if and only if $\mathcal{S} + \mathbf{v}$ is bounded and if and only if $\mathcal{S} - x$ is a bounded subset of $\mathcal{V}$.*

We note that the convex hull of a non-empty bounded subset $\mathcal{S}$ of $\mathbb{R}$ is included in the interval $[\inf \mathcal{S}, \sup \mathcal{S}]$ and hence is itself bounded. Using this fact, Def. 1, and the fact that flat functions preserve convex hulls (see Prop. 5 of Sect. 37) we obtain the following result.

Proposition 2: *The convex hull of every bounded subset of $\mathcal{E}$ is again bounded.*

Using Prop. 8 of Sect. 51 and Prop. 1 above, one immediately obtains the following result.

Proposition 3: *Every cell in $\mathcal{E}$ is bounded.*

Cell-Inclusion Theorem: *Let a point $q \in \mathcal{E}$ and a norming cell $\mathcal{B}$ in $\mathcal{V}$ be given. A subset $\mathcal{S}$ of $\mathcal{E}$ is bounded if and only if it is included in a cell modelled on $\mathcal{B}$ and centered at q.*

Proof: The "if" part follows from Prop. 3. Assume, then, that $\mathcal{S}$ is a bounded subset of $\mathcal{E}$. By Prop. 1, $\mathcal{S} - q$ is then a bounded subset of $\mathcal{V}$, so that $\boldsymbol{\lambda}_{>}(\mathcal{S} - q)$ is a bounded subset of $\mathbb{R}$ for all $\boldsymbol{\lambda} \in \mathcal{V}^*$. Choose a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ of $\mathcal{V}$ with dual $\mathbf{b}^*$. The sets $\mathbf{v} \in \mathcal{S} - q$ are bounded for all $i \in I$. Hence we may, and do, choose $h \in \mathbb{P}^{\times}$ such that $h > \max(|\mathbf{b}_i^*\mathbf{v}| \, |i \in I)$ for all $\mathbf{v} \in \mathcal{S} - q$. In view of (51.16), this means that

$$\mathrm{no}_{\mathrm{Box}(\mathbf{b})}(\mathbf{v}) < h \quad \text{for all} \quad \mathbf{v} \in \mathcal{S} - q$$

and hence

$$\mathcal{S} \subset q + h \, \mathrm{Box}(\mathbf{b}).$$

Therefore, $\mathcal{S}$ is included in a cell modelled on Box $(\mathbf{b})$ and centered at q. By Cor. 1 of the Norm-Equivalence Theorem of Sect. 51, $\mathcal{S}$ is also included in a cell modelled on $\mathcal{B}$ and centered at q.

Corollary 1: *Let $q \in \mathcal{E}$ and a norm ν on $\mathcal{V}$ be given. A subset $\mathcal{S}$ of $\mathcal{E}$ is bounded if and only if $\nu_{>}(\mathcal{S} - q)$ is a bounded subset of $\mathbb{P}$.*

The ν-**diameter** of a subset $\mathcal{S}$ of $\mathcal{E}$ is defined to be

$$\mathrm{diam}_{\nu}(\mathcal{S}) := \sup_{x,y \in \mathcal{S}} \nu(x - y) \in \overline{\mathbb{P}}. \tag{52.1}$$

Corollary 2: *Let ν be a norm on $\mathcal{V}$. A subset $\mathcal{S}$ of $\mathcal{E}$ is bounded if and only if it has finite ν-diameter.*

Let $\mathcal{V}$ and $\mathcal{V}'$ be linear spaces and let ν and ν' be norms on $\mathcal{V}$ and $\mathcal{V}'$, respectively. If $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$ then, by Prop. 3 and Prop. 1, $\mathbf{L}_{>}(\mathrm{Ce}(\nu))$ is a bounded subset of $\mathcal{V}'$. Hence, by Corollary 1 of the Cell-Inclusion Theorem, $\nu'_{>}(\mathbf{L}_{>}(\mathrm{Ce}(\nu)))$ is a bounded subset of $\mathbb{P}$ and therefore has a supremum in $\mathbb{P}$.

Definition 2: *If $\mathcal{V}, \mathcal{V}'$ are linear spaces, ν, ν' respective norms on $\mathcal{V}, \mathcal{V}'$, and $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$, then*

$$\|\mathbf{L}\|_{\nu,\nu'} := \sup \nu'_{>}(\mathbf{L}_{>}(\mathrm{Ce}(\nu))) \in \mathbb{P} \tag{52.2}$$

is called the **operator norm** of $\mathbf{L}$ *relative to* ν, ν'.

The following fact, easily verified, justifies the use of the term "norm" (see Def. 3 of Sect. 51).

Propositon 4: *The function* $(\mathbf{L} \mapsto \|\mathbf{L}\|_{\nu,\nu'}) : \mathrm{Lin}(\mathcal{V}, \mathcal{V}') \to \mathbb{P}$ *is a norm on* $\mathrm{Lin}(\mathcal{V}, \mathcal{V}')$.

Alternative descriptions of the operator norm are as follows.

Proposition 5: We have

$$\begin{aligned}
\|\mathbf{L}\|_{\nu,\nu'} &= \sup \nu'_{>}(\mathbf{L}_{>}(\overline{\mathrm{Ce}}(\nu))) && (52.3)\\
&= \sup \left\{ \frac{\nu'(\mathbf{L}\mathbf{v})}{\nu(\mathbf{v})} \mid \mathbf{v} \in \mathcal{V}^{\times} \right\} && (52.4)\\
&= \inf\{\sigma \in \mathbb{P} \mid \mathbf{L}_{>}(\mathrm{Ce}(\nu)) \subset \sigma \mathrm{Ce}(\nu')\} && (52.5)\\
&= \inf\{\sigma \in \mathbb{P} \mid \nu' \circ \mathbf{L} \le \sigma \nu\}, && (52.6)
\end{aligned}$$

so that

$$\nu'(\mathbf{L}\mathbf{v}) \le \|\mathbf{L}\|_{\nu,\nu'} \nu(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{52.7}$$

Repeated applications of (52.7) gives the following result.

Propositon 6: *If* $\mathcal{V}, \mathcal{V}', \mathcal{V}''$ *are linear spaces*, ν, ν', ν'' *respective norms*, $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$ *and* $\mathbf{M} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}'')$, *then*

$$\|\mathbf{ML}\|_{\nu,\nu''} \le \|\mathbf{M}\|_{\nu',\nu''} \|\mathbf{L}\|_{\nu,\nu'}. \tag{52.8}$$

If $\mathcal{V}$ is a linear space with norm ν, we write

$$\|\mathbf{L}\|_{\nu} := \|\mathbf{L}\|_{\nu,\nu} \quad \text{when} \quad \mathbf{L} \in \mathrm{Lin}(\mathcal{V}). \tag{52.9}$$

If $\mathcal{V}$ is non-zero, we have

$$\|\mathbf{1}_{\mathcal{V}}\|_{\nu} = 1 \tag{52.10}$$

no matter what ν is.

If $\mathcal{V}, \mathcal{V}'$ are non-zero linear spaces with respective norms ν, ν' and if $\mathbf{L} \in \mathrm{Lis}(\mathcal{V}, \mathcal{V}')$, then

$$\left(\|\mathbf{L}\|_{\nu,\nu'}\right)^{-1} \le \|\mathbf{L}^{-1}\|_{\nu',\nu}, \tag{52.11}$$

as is clear from (52.10) and Prop. 6.

Let $\mathcal{V}$ be a linear space and $\mathcal{V}^*$ its dual. If ν is a norm on $\mathcal{V}$, we define the **dual norm** ν^* on $\mathcal{V}^*$ so that $\nu^*(\boldsymbol{\lambda})$ is the operator norm of $\boldsymbol{\lambda} \in \mathcal{V}^* := \operatorname{Lin}(\mathcal{V}, \mathbb{R})$ relative to ν and the absolute-value-norm on $\mathbb{R}$. Thus, by Prop. 5, we have

$$\begin{aligned} \nu^*(\boldsymbol{\lambda}) &= \sup\{|\boldsymbol{\lambda}\mathbf{v}| \mid \mathbf{v} \in \operatorname{Ce}(\nu)\} && (52.12)\\ &= \sup\{|\boldsymbol{\lambda}\mathbf{v}| \mid \mathbf{v} \in \overline{\operatorname{Ce}}(\nu)\}. && (52.13)\\ &= \sup\left\{\frac{|\boldsymbol{\lambda}\mathbf{v}|}{\nu(\mathbf{v})} \mid \mathbf{v} \in \mathcal{V}^\times\right\}. && (52.14) \end{aligned}$$

The dual of the dual of a norm ν on $\mathcal{V}$ is a norm on $\mathcal{V}^{**} \simeq \mathcal{V}$. The following result states that this dual of the dual norm coincides with the original norm ν.

Norm-Duality Theorem: *If ν is a norm on the linear space $\mathcal{V}$, then $\nu = \nu^{**}$ when $\mathcal{V}$ is identified with $\mathcal{V}^{**}$. Specifically, we have*

$$\nu(\mathbf{v}) = \sup\{|\boldsymbol{\lambda}\mathbf{v}| \,\Big|\, \boldsymbol{\lambda} \in \operatorname{Ce}(\nu^*)\} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{52.15}$$

Proof: It follows from (52.14) that

$$|\boldsymbol{\lambda}\mathbf{v}| \le \nu^*(\boldsymbol{\lambda})\nu(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}, \boldsymbol{\lambda} \in \mathcal{V}^*. \tag{52.16}$$

Now let $\mathbf{v} \in \mathcal{V}$ be given. By (52.16) we have

$$\frac{|\boldsymbol{\lambda}\mathbf{v}|}{\nu^*(\boldsymbol{\lambda})} \le \nu(\mathbf{v}) \quad \text{for all} \quad \boldsymbol{\lambda} \in (\mathcal{V}^*)^\times.$$

Hence, using (52.14) with $\boldsymbol{\lambda}$ replaced by $\mathbf{v}, \mathcal{V}$ by $\mathcal{V}^*$, and ν by ν^* we get $\nu^{**}(\mathbf{v}) \le \nu(\mathbf{v})$.

If $\mathbf{v} = \mathbf{0}$ we have $\nu(\mathbf{v}) = 0 = \nu^{**}(\mathbf{v})$. If $\mathbf{v} \in \mathcal{V}^\times$, we apply Prop. 7 of Sect. 51 to $\mathcal{B} := \operatorname{Ce}(\nu)$ and $\mathcal{U} := \mathbb{R}\mathbf{v}$ to obtain a $\boldsymbol{\lambda} \in \mathcal{V}^*$ such that $\boldsymbol{\lambda}\mathbf{v} = \nu(\mathbf{v})$ and $|\boldsymbol{\lambda}\mathbf{u}| < 1$ for all $\mathbf{u} \in \operatorname{Ce}(\nu)$. In view of (52.12), we have $\nu^*(\boldsymbol{\lambda}) \le 1$ and hence $\boldsymbol{\lambda} \in \overline{\operatorname{Ce}}(\nu^*)$. Thus, using (52.13) with $\boldsymbol{\lambda}$ replaced by $\mathbf{v}$, $\mathcal{V}$ by $\mathcal{V}^*$, and ν by ν^*, we get $\nu^{**}(\mathbf{v}) \ge \boldsymbol{\lambda}\mathbf{v} = \nu(\mathbf{v})$ and hence $\nu^{**}(\mathbf{v}) = \nu(\mathbf{v})$. Since $\mathbf{v} \in \mathcal{V}$ was arbitrary, the conclusion follows. ■

Proposition 7: *If $\mathcal{V}, \mathcal{V}'$ are linear spaces with respective norms ν, ν' and if $\mathbf{L} \in \operatorname{Lin}(\mathcal{V}, \mathcal{V}')$, then*

$$\|\mathbf{L}^\top\|_{\nu'^*,\nu^*} = \|\mathbf{L}\|_{\nu,\nu'}. \tag{52.17}$$

Proof: Let $\boldsymbol{\mu} \in \mathcal{V}'^*$ be given. Applying Prop. 6 to the case when $\mathcal{V}'' := \mathbb{R}$, with absolute value as norm, and when $\mathbf{M}$ is replaced by $\boldsymbol{\mu}$, we get

$$\nu^*(\mathbf{L}^\top \boldsymbol{\mu}) = \nu^*(\boldsymbol{\mu}\mathbf{L}) \leq \|\mathbf{L}\|_{\nu,\nu'} \nu'^*(\boldsymbol{\mu}).$$

Using (52.6) with $\mathbf{L}$ replaced by $\mathbf{L}^\top$ we get $\|\mathbf{L}^\top\|_{\nu'^*,\nu^*} \leq \|\mathbf{L}\|_{\nu,\nu'}$. Applying this inequality to the case when $\mathbf{L}$ is replaced by $\mathbf{L}^\top$ and ν, ν' by ν'^*, ν^*, we also get

$$\|\mathbf{L}^{\top\top}\|_{\nu^{**},\nu'^{**}} \leq \|\mathbf{L}^\top\|_{\nu'^*,\nu^*}.$$

Since $\mathbf{L}^{\top\top} = \mathbf{L}$ and since $\nu = \nu^{**}$ and $\nu' = \nu'^{**}$ by Prop. 7, we obtain $\|\mathbf{L}\|_{\nu,\nu'} \leq \|\mathbf{L}^\top\|_{\nu'^*,\nu^*}$ and hence (52.17) ■

Examples:

1. The dual of a box-norm is easily seen to be a diamond-norm and vice-versa. Specifically, if $\mathbf{b}$ is a basis of $\mathcal{V}$ and $\mathbf{b}^*$ its dual basis, then

$$\left(\mathrm{no}_{\mathrm{Box}(\mathbf{b})}\right)^* = \mathrm{no}_{\mathrm{D\,md}(\mathbf{b}^*)}. \tag{52.18}$$

   (See Problem 6.)

2. If $\mathcal{V}$ has the structure of a genuine inner-product space, one can consider the magnitude norm $|\cdot|$ on $\mathcal{V}$. The dual of $|\cdot|$ is a norm on $\mathcal{V}^*$. Since $\mathcal{V}$ is identified with $\mathcal{V}^*$ by means of the inner product this dual norm can be viewed as a new norm on $\mathcal{V}$. Using the Inner-Product Inequality of Sect. 42, one easily proves that the dual of the magnitude is the magnitude itself.

3. Let $\mathcal{V}$ and $\mathcal{V}'$ be genuine inner-product spaces. The operator norm on $\mathrm{Lin}(\mathcal{V}, \mathcal{V}')$ relative to the magnitudes in $\mathcal{V}$ and $\mathcal{V}'$ is denoted simply by $\|\cdot\|$, so that

$$\|\mathbf{L}\| := \sup\left\{\frac{|\mathbf{L}\mathbf{v}|}{|\mathbf{v}|} \mid \mathbf{v} \in \mathcal{V}^\times\right\}. \tag{52.19}$$

   On the other hand, the space $\mathrm{Lin}(\mathcal{V}, \mathcal{V}')$ has a natural genuine inner product, defined by (44.5), and an associated magnitude-norm $|\cdot|$, defined by (44.13). This magnitude-norm $|\cdot|$ is distinct from the operator norm $|\cdot|$ unless $\mathcal{V}$ or $\mathcal{V}'$ have dimension one or zero. We will see in Sect. 8.4 that

$$|\mathbf{L}| \geq \|\mathbf{L}\| \geq \frac{1}{\sqrt{n}}|\mathbf{L}| \quad \text{for all} \quad \mathbf{L} \in \operatorname{Lin}(\mathcal{V}, \mathcal{V}') \tag{52.20}$$

when $n := \dim \mathcal{V} > 0$. ∎

Notes 52

(1) In the conventional treatments, boundedness is *defined* by the condition of the Cell-Inclusion Theorem. The concept obtained in this way is, at first view, a "boundedness *relative* to a norm (or norming cell) and a point". The boundedness is equivalent to the intrinsic concept of boundedness used here. This intrinsic concept corresponds to what is called "weak boundedness" in infinite-dimensional situations, where the two concepts are not equivalent.

53 Neighborhoods, Open and Closed Sets

We assume that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given.

Definition 1: *A subset of $\mathcal{E}$ is said to be a* **neighborhood** *(in $\mathcal{E}$) of a point $x \in \mathcal{E}$ if it includes a cell centered at x. The collection of all neighborhoods in $\mathcal{E}$ of x is denoted by* $\operatorname{Nhd}_x(\mathcal{E})$.

Of particular importance is the set $\operatorname{Nhd}_{\mathbf{0}}(\mathcal{V})$ of all neighborhoods in $\mathcal{V}$ of $\mathbf{0}$. It is clear that

$$\operatorname{Nhd}_x(\mathcal{E}) = \{x + \mathcal{N} \mid \mathcal{N} \in \operatorname{Nhd}_{\mathbf{0}}(\mathcal{V})\}. \tag{53.1}$$

The following results follow from Props.1, 2 of Sect.51.

Proposition 1: *If $\mathcal{N} \in \operatorname{Nhd}_{\mathbf{0}}(\mathcal{V})$ then $-\mathcal{N} \in \operatorname{Nhd}_{\mathbf{0}}(\mathcal{V})$ and $t\mathcal{N} \in \operatorname{Nhd}_{\mathbf{0}}(\mathcal{V})$ for all $t \in \mathbb{R}^{\times}$.*

Proposition 2: *Every subset of $\mathcal{E}$ that includes a neighborhood of x is a neighborhood of x. The intersection of a finite collection of neighborhoods of x is a neighborhood of x.*

Using Cor.1 and Cor.2 of the Norm-Equivalence Theorem of Sect.51, one immediately obtains the following result.

Proposition 3: *Let $\mathcal{B}$ be any norming cell in $\mathcal{V}$. A subset of $\mathcal{E}$ is a neighborhood of a point in $\mathcal{E}$ if and only if it includes a cell modelled on $\mathcal{B}$ and centered at that point.*

Let ν be any norm on $\mathcal{V}$. A subset of $\mathcal{E}$ is a neighborhood of a point in $\mathcal{E}$ if and only if it includes a ν-cell centered at that point.

Definition 2: *Let $\mathcal{S}$ be a subset of $\mathcal{E}$. We say that the point $x \in \mathcal{E}$ is* **interior** *to $\mathcal{S}$ (in $\mathcal{E}$) if $\mathcal{S} \in \operatorname{Nhd}_x(\mathcal{E})$. We say that $x \in \mathcal{E}$ is* **close** *to $\mathcal{S}$ (in $\mathcal{E}$) if $\mathcal{S}$ meets every neighborhood of x, i.e. if $\mathcal{N} \cap \mathcal{S} \neq \emptyset$ for all $\mathcal{N} \in \operatorname{Nhd}_x(\mathcal{E})$.*

The **interior** *and the* **closure** *of $\mathcal{S}$ are defined to be the set of all points interior to $\mathcal{S}$ and close to $\mathcal{S}$, respectively. They are denoted by*

$$\operatorname{Int}\mathcal{S} := \{x \in \mathcal{E} \mid \mathcal{S} \in \mathrm{Nhd}_x(\mathcal{E})\}, \tag{53.2}$$

$$\operatorname{Clo}\mathcal{S} := \{x \in \mathcal{E} \mid \mathcal{N} \cap \mathcal{S} \neq \emptyset \quad \text{for all} \quad \mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})\}. \tag{53.3}$$

We will see later (Prop.12) that the definition of "closure" just given does not clash with the one given in Defs.2 and 4 of Sect.51.

It is evident that

$$\operatorname{Int}\mathcal{S} \subset \mathcal{S} \subset \operatorname{Clo}\mathcal{S}. \tag{53.4}$$

Also, the operations Int and Clo are isotone with respect to inclusion, i.e. we have

$$\mathcal{S} \subset \mathcal{T} \implies \operatorname{Int}\mathcal{S} \subset \operatorname{Int}\mathcal{T} \quad \text{and} \quad \operatorname{Clo}\mathcal{S} \subset \operatorname{Clo}\mathcal{T} \tag{53.5}$$

for all subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{E}$.

The interior and closure of a set $\mathcal{S}$ depend, on the face of it, not only on $\mathcal{S}$ but also on the flat space $\mathcal{E}$. We write $\operatorname{Int}_{\mathcal{E}}\mathcal{S}$ and $\operatorname{Clo}_{\mathcal{E}}\mathcal{S}$ if we wish to make this dependence explicit. We shall see at the end of this section that the closure is, in fact, independent of what one considers to be the flat space that includes $\mathcal{S}$. More precisely, if $\mathcal{S}$ is included in a proper flat $\mathcal{F}$ in $\mathcal{E}$, then $\operatorname{Clo}_{\mathcal{F}}\mathcal{S} = \operatorname{Clo}_{\mathcal{E}}\mathcal{S}$. In this case $\operatorname{Int}_{\mathcal{E}}\mathcal{S}$ is necessarily empty, but $\operatorname{Int}_{\mathcal{F}}\mathcal{S}$ need not be.

The following are easily seen to be valid for every subset $\mathcal{S}$ and $\mathcal{E}$:

$$\operatorname{Int}\mathcal{S} \neq \emptyset \implies \operatorname{Fsp}\mathcal{S} = \mathcal{E}, \tag{53.6}$$

$$\operatorname{Int}\operatorname{Clo}\mathcal{S} \supset \operatorname{Int}\mathcal{S}, \tag{53.7}$$

$$\operatorname{Clo}\operatorname{Int}\mathcal{S} \subset \operatorname{Clo}\mathcal{S}. \tag{53.8}$$

In view of (53.1) and the definitions (53.2), (53.3) we have

$$\operatorname{Int}(x + \mathcal{R}) = x + \operatorname{Int}\mathcal{R}, \quad \operatorname{Clo}(x + \mathcal{R}) = x + \operatorname{Clo}\mathcal{R} \tag{53.9}$$

for all $x \in \mathcal{E}$ and all subsets $\mathcal{R}$ of $\mathcal{V}$.

Proposition 4: *The complement of the closure [interior] of a set is the interior [closure] of its complement, i.e.*

$$\mathcal{E} \setminus \operatorname{Clo}\mathcal{S} = \operatorname{Int}(\mathcal{E} \setminus \mathcal{S}), \tag{53.10}$$

$$\mathcal{E} \setminus \operatorname{Int}\mathcal{S} = \operatorname{Clo}(\mathcal{E} \setminus \mathcal{S}) \tag{53.11}$$

hold for all subsets $\mathcal{S}$ of $\mathcal{E}$.

Proof: Note that $\mathcal{S} \cap \mathcal{N} = \emptyset$ holds for a subset $\mathcal{N}$ of $\mathcal{E}$ if and only if $\mathcal{N} \subset \mathcal{E} \setminus \mathcal{S}$. By the first statement of Prop.2, it follows that

$$(\mathcal{S} \cap \mathcal{N} = \emptyset \quad \text{for some} \quad \mathcal{N} \in \operatorname{Nhd}_x(\mathcal{E})) \iff \mathcal{E} \setminus \mathcal{S} \in \operatorname{Nhd}_x(\mathcal{E}).$$

Therefore, for every $x \in \mathcal{E}$, we have

$$\begin{aligned} x \in \operatorname{Int}(\mathcal{E} \setminus \mathcal{S}) \quad &\iff \quad \mathcal{E} \setminus \mathcal{S} \in \operatorname{Nhd}_x(\mathcal{E}) \\ &\iff \quad (\mathcal{S} \cap \mathcal{N} = \emptyset \quad \text{for some} \quad \mathcal{N} \in \operatorname{Nhd}_x(\mathcal{E})) \\ &\iff \quad x \notin \operatorname{Clo}\mathcal{S} \\ &\iff \quad x \in \mathcal{E} \setminus \operatorname{Clo}\mathcal{S}, \end{aligned}$$

which proves (53.10). To obtain (53.11), we merely need to substitute $\mathcal{E} \setminus \mathcal{S}$ for $\mathcal{S}$ in (53.10) and take the complement. ■

The **boundary** of a subset $\mathcal{S}$ of $\mathcal{E}$ is defined to be

$$\operatorname{Bdy}\mathcal{S} := \operatorname{Clo}\mathcal{S} \setminus \operatorname{Int}\mathcal{S}. \tag{53.12}$$

We say that $\mathcal{S} \subset \mathcal{E}$ is **open** if $\operatorname{Int}\mathcal{S} = \mathcal{S}$ and **closed** if $\operatorname{Clo}\mathcal{S} = \mathcal{S}$. We will see later (Prop.13) that these definitions of "open" and "boundary" do not clash with the ones given in Sect.38. The following results are consequences of Prop.4 and Prop.2.

Proposition 5: *A subset of $\mathcal{E}$ is open [closed] if and only if its complement in $\mathcal{E}$ is closed [open].*

Proposition 6: *The union of any collection of open sets is open. The intersection of any collection of closed sets is closed. The intersection of a finite collection of open sets is open. The union of a finite collection of closed sets is closed.*

Proposition 7: *If $\mathcal{D}$ is an open subset and $\mathcal{G}$ a closed subset of $\mathcal{E}$, then $\mathcal{D} \setminus \mathcal{G}$ is open and $\mathcal{G} \setminus \mathcal{D}$ closed.*

The empty set $\emptyset$ and the whole space $\mathcal{E}$ are both open and closed. Let $q \in \mathcal{E}$. In view of (53.9), a set $\mathcal{S}$ is open [closed] in $\mathcal{E}$ if and only if $\mathcal{S} - q$ is open [closed] in $\mathcal{V}$.

Proposition 8: *Every cell is open.*

Proof: In view of Prop.5 of Sect.51, it is sufficient to show that for every norm ν on $\mathcal{V}$, the norming cell $\operatorname{Ce}(\nu)$ is open. Let $\mathbf{v} \in \operatorname{Ce}(\nu)$. Then $0 \le \nu(\mathbf{v}) < 1$ and hence

$$r := 1 - \nu(\mathbf{v}) \in \,]0, 1].$$

If $\mathbf{u} \in \mathbf{v} + r\,\mathrm{Ce}(\nu)$ then $\mathbf{v} - \mathbf{u} \in r\,\mathrm{Ce}(\nu)$ and hence $\nu(\mathbf{v} - \mathbf{u}) < r$. Using the subaddivity property (N_2) of the Def.3, Sect.51, we find that

$$\nu(\mathbf{u}) \leq \nu(\mathbf{v} - \mathbf{u}) + \nu(\mathbf{v}) < r + \nu(\mathbf{v}) = 1,$$

i.e. that $\mathbf{u} \in \mathrm{Ce}(\nu)$. Therefore $\mathrm{Ce}(\nu)$ includes the cell $\mathbf{v} + r\,\mathrm{Ce}(\nu)$ centered at $\mathbf{v}$, which shows that $\mathrm{Ce}(\nu) \in \mathrm{Nhd}_{\mathbf{v}}(\mathcal{V})$. Since $\mathbf{v} \in \mathrm{Ce}(\nu)$ was arbitrary, it follows that $\mathrm{Ce}(\nu)$ is open. ∎

Proposition 9: *The interior of every set is open and its closure is closed.*

Proof: Let $\mathcal{S}$ be a subset of $\mathcal{E}$ and let $x \in \mathrm{Int}\,\mathcal{S}$, so that $\mathcal{S} \in \mathrm{Nhd}_x(\mathcal{E})$. By Def.1, $\mathcal{S}$ includes a cell $\mathcal{C}$ centered at x. Since $\mathcal{C}$ is open by Prop.8, $\mathcal{C}$ and hence $\mathcal{S}$ is a neighborhood of every point in $\mathcal{C}$, which means that $\mathcal{C} \subset \mathrm{Int}\,\mathcal{S}$. It follows that $\mathrm{Int}\,\mathcal{S}$ is a neighborhood of x. Since $x \in \mathrm{Int}\,\mathcal{S}$ was arbitrary, it follows that $\mathrm{Int}\,\mathcal{S}$ is open.

The proof just given and (53.10) show that $\mathcal{E} \setminus \mathrm{Clo}\,(\mathcal{S}) = \mathrm{Int}\,(\mathcal{E} \setminus \mathcal{S})$ is open. Hence, by Prop.5, $\mathrm{Clo}\,\mathcal{S}$ is closed. ∎

It follows from Props.6 and 9 that the closure-mapping $\mathrm{Clo} : \mathrm{Sub}\,\mathcal{E} \to \mathrm{Sub}\,\mathcal{E}$, which assigns to each subset of $\mathcal{E}$ its closure, is nothing but the span-mapping corresponding to the (intersection-stable) collection of all closed subsets of $\mathcal{E}$ (see Sect.03). Using (03.26) and Prop.4 one obtains

$$\mathrm{Clo}\,(\mathrm{Clo}\,\mathcal{S}) = \mathrm{Clo}\,\mathcal{S} \quad \text{and} \quad \mathrm{Int}\,(\mathrm{Int}\,\mathcal{S}) = \mathrm{Int}\,\mathcal{S} \tag{53.13}$$

for all $\mathcal{S} \in \mathrm{Sub}\,\mathcal{E}$. Moreover, we have

Proposition 10: *The interior of a set is the largest (with respect to inclusion) among the open sets included in it. The closure of a set is the smallest (with respect to inclusion) among the closed sets that include it.*

Applying Prop.9 and Prop.7 to (53.12), we find

Proposition 11: *The boundary of every set is closed.*

The following result shows that the closure of a norming cell in the sense of Def.2 above coincides with its closure in the sense of Def.2 of Sect.51.

Proposition 12: *For every norm ν on $\mathcal{V}$, we have*

$$\mathrm{Clo}\,\mathrm{Ce}(\nu) = \overline{\mathrm{Ce}}(\nu) = \nu^{<}(\,[\,0, 1\,]\,) \tag{53.14}$$

and

$$\mathrm{Bdy}\,\mathrm{Ce}(\nu) = \nu^{<}(\{1\}). \tag{53.15}$$

Hence, for every norming cell $\mathcal{B}$ of $\mathcal{V}$, we have $\bar{\mathcal{B}} = \mathrm{Clo}\,\mathcal{B}$.

Proof: Let $\mathbf{v} \in \overline{\mathrm{Ce}}(\nu)$ and $\mathcal{N} \in \mathrm{Nhd}_{\mathbf{v}}(\mathcal{V})$ be given. By Prop.3, there is a $\sigma \in]0,1]$ such that $\mathbf{v} + \sigma\mathrm{Ce}(\nu) \subset \mathcal{N}$. Since $-\frac{1}{2}\mathbf{v} \in \mathrm{Ce}(\nu)$, it follows that $(1-\frac{\sigma}{2})\mathbf{v} = \mathbf{v} + \sigma(-\frac{1}{2}\mathbf{v}) \in \mathcal{N}$. Since $1 - \frac{\sigma}{2} \in [0,1[$ we also have $(1-\frac{\sigma}{2})\mathbf{v} \in \mathrm{Ce}(\nu)$ and hence $(1-\frac{\sigma}{2})\,\mathbf{v} \in \mathcal{N} \cap \mathrm{Ce}(\nu)$, showing that $\mathcal{N} \cap \mathrm{Ce}(\nu) \neq \emptyset$. Since $\mathcal{N} \in \mathrm{Nhd}_{\mathbf{v}}(\mathcal{V})$ was arbitrary, it follows that $\mathbf{v} \in \mathrm{Clo}\,\mathrm{Ce}(\nu)$.

Assume, now, that $\mathbf{v} \notin \overline{\mathrm{Ce}}(\nu)$ and hence that $\nu(\mathbf{v}) > 1$. Put $r := \nu(\mathbf{v}) - 1$. If $\mathbf{u} \in \mathbf{v} + r\mathrm{Ce}(\nu)$ then $\nu(\mathbf{u}-\mathbf{v}) < r$ and hence $\nu(\mathbf{u}) \geq \nu(\mathbf{v}) - \nu(\mathbf{v}-\mathbf{u}) > \nu(\mathbf{v}) - r = 1$, i.e. $\mathbf{u} \notin \overline{\mathrm{Ce}}(\nu)$. Therefore $(\mathbf{v} + r\,\mathrm{Ce}(\nu)) \cap \mathrm{Ce}(\nu) = \emptyset$. Since $\mathbf{v} + r\,\mathrm{Ce}(\nu) \in \mathrm{Nhd}_{\mathbf{v}}(\mathcal{V})$, this shows that $\mathbf{v} \notin \mathrm{Clo}\,\mathrm{Ce}(\nu)$.

We have proved that $\mathbf{v} \in \overline{\mathrm{Ce}}(\nu)$ if and only if $\mathbf{v} \in \mathrm{Clo}\,\mathrm{Ce}(\nu)$, which means that (53.14) holds. (53.15) follows from (53.14) and (53.12). ■

The following result shows that the boundary of a half-space in the sense of Sect.38 coincides with its boundary in the sense of (53.12) and that an open-half-space in the sense of Sect.38 is an open set as defined above.

Proposition 13: *Let a be a non-constant flat function on $\mathcal{E}$. Then the half-space $a^{<}(\mathbb{P})$ is closed, its interior is $a^{<}(\mathbb{P}^{\times})$, and its boundary is $a^{<}(\{0\})$.*

Proof: Let $x \in a^{<}(\mathbb{P}^{\times})$ be given. Consider the "strip"

$$\mathcal{S} := \frac{1}{a(x)}(\nabla a)^{<}(\,]-1,1[\,) = \{\mathbf{v} \in \mathcal{V} \;\Big|\; |(\nabla a)\mathbf{v}\,| < a(x)\}.$$

Using the fact that $\nabla a \neq 0$, one easily shows that if $\mathcal{B}$ is a norming cell, so is $\mathcal{B} \cap \mathcal{S}$. On the other hand, we have $x + \mathcal{B} \cap \mathcal{S} \subset a^{<}(\mathbb{P}^{\times})$, which proves that $a^{<}(\mathbb{P}^{\times}) \in \mathrm{Nhd}_x(\mathcal{E})$. Since $x \in a^{<}(\mathbb{P}^{\times})$ was arbitrary, it follows that $a^{<}(\mathbb{P}^{\times})$ is open. Applying this result to $-a$ instead of a and using Prop.5, we find that $(-a)^{<}(\mathbb{P}^{\times}) = a^{<}(-\mathbb{P}^{\times}) = \mathcal{E} \setminus a^{<}(\mathbb{P})$ is open and hence that $a^{<}(\mathbb{P})$ is closed.

Now let $x \in a^{<}(\{0\})$ be given. Since $\nabla a \neq 0$, we may choose a $\mathbf{v} \in \mathcal{V}$ such that $(\nabla a)\mathbf{v} > 0$. Then the intersection of the half-space $a^{<}(\mathbb{P})$ and the line $x + \mathbb{R}\mathbf{v}$ is $x + \mathbb{P}\mathbf{v}$, which includes no open segment centered at x. Hence $a^{<}(\mathbb{P})$ cannot include a cell centered at x, which implies that $x \notin \mathrm{Int}\,a^{<}(\mathbb{P})$. Since $x \in a^{<}(\{0\})$ was arbitrary, we conclude that $a^{<}(\{0\}) \cap \mathrm{Int}\,a^{<}(\mathbb{P}) = \emptyset$. Since $a^{<}(\mathbb{P}^{\times})$ is open and $a^{<}(\mathbb{P}) = a^{<}(\mathbb{P}^{\times}) \cup a^{<}(\{0\})$, we see that $\mathrm{Int}\,a^{<}(\mathbb{P}) = a^{<}(\mathbb{P}^{\times})$ and $\mathrm{Bdy}\,a^{<}(\mathbb{P}) = a^{<}(\{0\})$. ■

Let $\mathcal{F}$ be a flat in $\mathcal{E}$ and let x be a point in $\mathcal{F}$. At the end of Sect.51 we saw that a subset of $\mathcal{F}$ is a cell in $\mathcal{F}$ centered at x if and only if it is the intersection of $\mathcal{F}$ and a cell in $\mathcal{E}$ centered at x. It follows from Def.1 that the set of all neighborhoods of x in $\mathcal{F}$ is given by

$$\mathrm{Nhd}_x(\mathcal{F}) = \{\mathcal{N} \cap \mathcal{F} \mid \mathcal{N} \in \mathrm{Nhd}_x\mathcal{E}\}. \tag{53.16}$$

Now let $\mathcal{S}$ be a subset of $\mathcal{F}$. Using (53.16) and the definition (53.3) of closure, we see that $\mathrm{Clo}_{\mathcal{F}}(\mathcal{S}) = \mathcal{F} \cap \mathrm{Clo}\,(\mathcal{S})$. On the other hand, it follows from Props.11 and 13 that hyperplanes in $\mathcal{E}$ are closed. Hence, by Prop.6 above and Prop.5 of Sect.36 every flat $\mathcal{F}$ is closed in $\mathcal{E}$. We conclude that $\mathrm{Clo}_{\mathcal{F}}(\mathcal{S}) = \mathrm{Clo}\,(\mathcal{S})$, as mentioned earlier.

Let $\mathcal{E}_1, \mathcal{E}_2$ be flat spaces with translation spaces $\mathcal{V}_1, \mathcal{V}_2$ with norms ν_1, ν_2, respectively. Using Prop.3 and (51.23), we obtain the following result.

Proposition 14: *A subset of $\mathcal{E}_1 \times \mathcal{E}_2$ is a neighborhood (in $\mathcal{E}_1 \times \mathcal{E}_2$) of a point $(x_1, x_2) \in \mathcal{E}_1 \times \mathcal{E}_2$ if and only if it includes the set-product of a ν_1-cell centered at x_1 and a ν_2-cell centered at x_2.*

The following result is an easy consequence of Prop.14 and the definitions given earlier.

Proposition 15: *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subsets of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Then*

$$\mathrm{Clo}\,(\mathcal{S}_1 \times \mathcal{S}_2) = \mathrm{Clo}\,\mathcal{S}_1 \times \mathrm{Clo}\,\mathcal{S}_2, \tag{53.17}$$

$$\mathrm{Int}\,(\mathcal{S}_1 \times \mathcal{S}_2) = \mathrm{Int}\,\mathcal{S}_1 \times \mathrm{Int}\,\mathcal{S}_2, \tag{53.18}$$

$$\mathrm{Bdy}\,(\mathcal{S}_1 \times \mathcal{S}_2) = (\mathrm{Bdy}\,\mathcal{S}_1 \times \mathrm{Clo}\,\mathcal{S}_2) \cup (\mathrm{Clo}\,\mathcal{S}_1 \times \mathrm{Bdy}\,\mathcal{S}_2). \tag{53.19}$$

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are open [closed, bounded] then $\mathcal{S}_1 \times \mathcal{S}_2$ is an open [closed, bounded] subset of $\mathcal{E}_1 \times \mathcal{E}_2$.

Notes 53

(1) The term "neighborhood" is sometimes used (especially in the older literature) for what we call "open neighborhood".

(2) Other notations for the closure $\mathrm{Clo}\,\mathcal{S}$ of a set $\mathcal{S}$ are $\overline{\mathcal{S}}, \mathcal{S}^-, \mathcal{S}^c$, and $\mathrm{cl}\mathcal{S}$.

(3) Other notations for the interior $\mathrm{Int}\,\mathcal{S}$ of a set $\mathcal{S}$ are $\mathring{\mathcal{S}}, \mathcal{S}^\circ$, and $\mathrm{int}\,\mathcal{S}$.

(4) The boundary $\mathrm{Bdy}\,\mathcal{S}$ of a set is very often denoted by $\partial\mathcal{S}$.

(5) A few people use the term "frontier" and the notation $\mathrm{fr}\mathcal{S}$ for what we call the boundary $\mathrm{Bdy}\,\mathcal{S}$ of a set $\mathcal{S}$.

54 Topology of Convex Sets

The following two results show that the conclusion of the Half-Space-Inclusion Theorem can be strengthened for open and for closed convex sets.

Proposition 1: *Let $\mathcal{C}$ be a non-empty open convex subset of a flat space $\mathcal{E}$ and let $z \in \mathcal{E} \setminus \mathcal{C}$. Then there is an open-half-space that includes $\mathcal{C}$ and* *has z on its boundary. In other words, there is an $a \in (\mathrm{Flf}_z(\mathcal{E}))^\times$ such that $a_>(\mathcal{C}) \subset \mathbb{P}^\times$.*

Proof: By the Half-Space Inclusion Theorem we can find $a \in (\mathrm{Flf}_z(\mathcal{E}))^\times$ such that $a|_\mathcal{C} \geq 0$, i.e. $\mathcal{C} \subset a^<(\mathbb{P})$. Since $\mathcal{C}$ is open and $\mathrm{Int}\, a^<(\mathbb{P}) = a^<(\mathbb{P}^\times)$ by Prop.13 of Sect.53, it follows from Prop.10 of Sect.53 that $\mathcal{C} \subset a^<(\mathbb{P}^\times)$, i.e. $a_>(\mathcal{C}) \subset \mathbb{P}^\times$. ■

Proposition 2: *Let $\mathcal{C}$ be a non-empty convex subset of a flat space $\mathcal{E}$ and let $z \in \mathcal{E} \setminus \mathrm{Clo}\,(\mathcal{C})$. Then there is a half-space that includes $\mathcal{C}$ and does not contain z. Equivalently, there is an $a \in \mathrm{Flf}(\mathcal{E})$ such that $a|_\mathcal{C} \geq 0$ and $a(z) = -1$.*

Proof: By (53.10) we have $z \in \mathrm{Int}\,(\mathcal{E} \setminus \mathcal{C})$, i.e. $\mathcal{E} \setminus \mathcal{C} \in \mathrm{Nhd}_z(\mathcal{E})$. Hence we may choose a norming cell $\mathcal{B}$ such that $z + \mathcal{B} \subset \mathcal{E} \setminus \mathcal{C}$, which is equivalent to $z \notin \mathcal{C} - \mathcal{B}$. The set $\mathcal{C} - \mathcal{B} = \mathcal{C} + \mathcal{B}$ is convex by Prop.4 of Sect.37 and non-empty because both $\mathcal{C}$ and $\mathcal{B}$ are non-empty. By the Half-Space Inclusion Theorem there is a $b \in (\mathrm{Flf}_z(\mathcal{E}))^\times$ such that $\mathcal{C} - \mathcal{B} \subset b^<(\mathbb{P})$. Since z belongs to the closure $b^<(\mathbb{P})$ of $b^<(\mathbb{P}^\times)$ (see Prop.13 of Sect.53), the neighborhood $z + \mathcal{B}$ of z must intersect $b^<(\mathbb{P}^\times)$. Hence we may, and do, choose a $\mathbf{u} \in \mathcal{B}$ such that $z + \mathbf{u} \in b^<(\mathbb{P}^\times)$, i.e. $0 < b(z + \mathbf{u}) = b(z) + (\nabla b)\mathbf{u} = (\nabla b)\mathbf{u}$. Since $\mathcal{C} - \mathbf{u} \subset b^<(\mathbb{P})$, it follows that $0 \leq b(x - \mathbf{u}) = b(x) - (\nabla b)\mathbf{u}$ for all $x \in \mathcal{C}$ and hence $(b - (\nabla b)\mathbf{u})|_\mathcal{C} \geq 0$. On the other hand, $(b - (\nabla b)\mathbf{u})(z) = -(\nabla b)\mathbf{u} < 0$. Hence $a := ((\nabla b)\mathbf{u})^{-1}(b - (\nabla b)\mathbf{u})$ has the desired properties. ■

Prop.2 is the key to the proof the following basic result.

Half-Space Intersection Theorem: *Let $\mathcal{S}$ be a non-empty subset of a flat space $\mathcal{E}$. Then the closure of the convex hull of $\mathcal{S}$ is the intersection of all half-spaces that include $\mathcal{S}$, i.e.*

$$\mathrm{Clo}\,\mathrm{Cxh}\mathcal{S} = \bigcap \{a^<(\mathbb{P}) \mid a \in \mathrm{Flf}(\mathcal{E}),\ \mathcal{S} \subset a^<(\mathbb{P})\}. \tag{54.1}$$

Proof: Put $\mathcal{A} := \{a \in \mathrm{Flf}(\mathcal{E}) \mid \mathcal{S} \subset a^<(\mathbb{P})\}$ and $\mathcal{C} := \bigcap(a^<(\mathbb{P}) \mid a \in \mathcal{A})$, so that $\mathcal{S} \subset a^<(\mathbb{P})$ for all $a \in \mathcal{A}$ and hence $\mathcal{S} \subset \mathcal{C}$. Since $a^<(\mathbb{P})$ is convex and closed (see Prop.13 of Sect.53), it follows that $\mathcal{C}$ is convex and closed (see Prop.1 of Sect.37 and Prop.6 of Sect.53). Therefore, we must have $\mathrm{Clo}\,\mathrm{Cxh}\mathcal{S} \subset \mathcal{C}$.

Now let $z \in \mathcal{E} \setminus \mathrm{Clo}\,\mathrm{Cxh}\mathcal{S}$ be given. By Prop.2, we may choose $a \in \mathrm{Flf}(\mathcal{E})$ such that $a \in \mathcal{A}$ but $a(z) = -1$, so that $z \notin a^<(\mathbb{P})$. It follows that $z \notin \mathcal{C}$, i.e. $z \in \mathcal{E} \setminus \mathcal{C}$. Since $z \in \mathcal{E} \setminus \mathrm{Clo}\,\mathrm{Cxh}\mathcal{S}$ was arbitrary, it follows that $\mathcal{E} \setminus \mathrm{Clo}\,\mathrm{Cxh}\mathcal{S} \subset \mathcal{E} \setminus \mathcal{C}$, and hence $\mathcal{C} \subset \mathrm{Clo}\,\mathrm{Cxh}\mathcal{S}$. ■

Applying the Theorem to closed convex sets we obtain:

Corollary 1: *Every closed convex set is the intersection of all half-spaces that include it.*

Since half-spaces are convex and intersections of collections of convex sets are convex, we have the following consequence of the Theorem.

Corollary 2: *The closure of every convex set is again convex.*

Corollary 3: *If the subset $\mathcal{S}$ of $\mathcal{E}$ is bounded, so is* $\operatorname{Clo}\operatorname{Cxh}\mathcal{S}$.

Proof: The assertion is trivial when $\mathcal{S}$ is empty. We may assume, therefore that $\mathcal{S}$ is not empty and bounded. Let $b \in \operatorname{Flf}(\mathcal{E})$ be given. Since $\mathcal{S}$ is bounded, we have $\xi := \sup b_>(\mathcal{S}) < \infty$ and hence $\mathcal{S} \subset (\xi - b)^<(\mathbb{P})$. By (54.1), it follows that $\operatorname{Clo}\operatorname{Cxh}\mathcal{S} \subset (\xi - b)^<(\mathbb{P})$, and hence that $b_>(\operatorname{Clo}\operatorname{Cxh}\mathcal{S})$ has ξ as an upper bound. Replacing b by $-b$ in the argument above we see that $b_>(\operatorname{Clo}\operatorname{Cxh}\mathcal{S})$ has also a lower bound and hence is a bounded subset of $\mathbb{R}$. Since $b \in \operatorname{Flf}(\mathcal{E})$ was arbitrary, $\operatorname{Clo}\operatorname{Cxh}\mathcal{S}$ is bounded. ∎

Remark: Another proof of Corollary 3 is based on the Cell-Inclusion Theorem of Sect.52: If $\mathcal{S}$ is bounded, then we can choose $q \in \mathcal{E}$ and a norming cell $\mathcal{B}$ such that $\mathcal{S} \subset q + \mathcal{B}$. Since $q + \mathcal{B}$ is convex we have $\operatorname{Cxh}\mathcal{S} \subset q + \mathcal{B}$ and hence $\operatorname{Clo}\operatorname{Cxh}\mathcal{S} \subset q + \operatorname{Clo}\mathcal{B}$. Using Prop.12 of Sect.53, it follows that $\operatorname{Clo}\operatorname{Cxh}\mathcal{S} \subset q + \bar{\mathcal{B}} \subset q + 2\mathcal{B}$. Since $q + 2\mathcal{B}$ is again a cell, the Cell-Inclusion Theorem tells us that $\operatorname{Clo}\operatorname{Cxh}\mathcal{S}$ is bounded. ∎

Proposition 3: *Let $\mathcal{C}$ be a convex set and let $x \in \operatorname{Int}\mathcal{C}$, $y \in \operatorname{Clo}\mathcal{C}$ be given such that $x \neq y$. Then $]x, y[\subset \operatorname{Int}\mathcal{C}$.*

Proof: Let $z \in]x, y[$ be given. By (51.1), this means that $z = \lambda x + \mu y$ for some $\lambda, \mu \in \mathbb{P}^\times$ with $\lambda + \mu = 1$. Since $x \in \operatorname{Int}\mathcal{C}$, we have $\mathcal{C} \in \operatorname{Nhd}_x(\mathcal{E})$. If we put $\mathcal{B} := \mathcal{C} - x$, it follows that $\mathcal{B} \in \operatorname{Nhd}_{\mathbf{0}}(\mathcal{V})$ and hence, by Prop.1 of Sect.51, $-\lambda\mathcal{B},\ \lambda^2\mathcal{B} \in \operatorname{Nhd}_0(\mathcal{V})$. Since $y \in \operatorname{Clo}\mathcal{C}$ and $y - \lambda\mathcal{B} \in \operatorname{Nhd}_y(\mathcal{E})$, we have $(y - \lambda\mathcal{B}) \cap \mathcal{C} \neq \emptyset$, and we may therefore choose $\mathbf{w} \in \mathcal{B}$ such that $y - \lambda\mathbf{w} \in \mathcal{C}$.

Now let $\mathbf{u} \in \mathcal{B}$ be given. Since $\lambda^2 + \lambda\mu + \mu = \lambda(\lambda + \mu) + \mu = 1$ we can form the convex combination of $(x + \mathbf{u}, x + \mathbf{w}, y - \lambda\mathbf{w})$ at $(\lambda^2, \lambda\mu, \mu)$: $\lambda^2(x + \mathbf{u}) + \lambda\mu(x + \mathbf{w}) + \mu(y - \lambda\mathbf{w}) = z + \lambda^2\mathbf{u}$. Since $\mathcal{C}$ is convex and since $x + \mathbf{u}, x + \mathbf{w}, y - \lambda\mathbf{w} \in \mathcal{C}$, it follows from Prop.5 of Sect.37 that $z + \lambda^2\mathbf{u} \in \mathcal{C}$. Since $\mathbf{u} \in \mathcal{B}$ was arbitrary, we conclude that $z + \lambda^2\mathcal{B} \subset \mathcal{C}$. Since $z + \lambda^2\mathcal{B} \in \operatorname{Nhd}_z(\mathcal{E})$, it follows that $z \in \operatorname{Int}\mathcal{C}$. Since $z \in]x, y[$ was arbitrary, we get the desired conclusion. ∎

Since $\operatorname{Int}\mathcal{C} \subset \operatorname{Clo}\mathcal{C}$ for every set $\mathcal{C}$, Prop.3 has the following immediate consequence.

Proposition 4: *The interior of every convex set is convex.*

The following result states that for convex sets with non-empty interiors the inclusions (53.7) and (53.8) can be reversed.

Proposition 5: *If $\mathcal{C}$ is convex and $\operatorname{Int}\mathcal{C} \neq \emptyset$ then*

$$\operatorname{Clo}\operatorname{Int}\mathcal{C} = \operatorname{Clo}\mathcal{C}, \quad \operatorname{Int}\operatorname{Clo}\mathcal{C} = \operatorname{Int}\mathcal{C}. \tag{54.2}$$

Proof: Since $\operatorname{Int}\mathcal{C} \neq \emptyset$, we may and do choose a point $q \in \operatorname{Int}\mathcal{C}$. Let $x \in \operatorname{Clo}\mathcal{C}$ be given. If $x \neq q$ we have, by Prop.3, $]q,x[\subset \operatorname{Int}\mathcal{C}$. Since every cell centered at x has a non-empty intersection with $]q,x[$ and hence $\operatorname{Int}\mathcal{C}$, we conclude that $x \in \operatorname{Clo}\operatorname{Int}\mathcal{C}$. Since $q \in \operatorname{Int}\mathcal{C} \subset \operatorname{Clo}\operatorname{Int}\mathcal{C}$, we see that $x \in \operatorname{Clo}\operatorname{Int}\mathcal{C}$ holds for all $x \in \operatorname{Clo}\mathcal{C}$ and hence that $\operatorname{Clo}\mathcal{C} \subset \operatorname{Clo}\operatorname{Int}\mathcal{C}$, which proves $(54.2)_1$.

Now let $y \in \operatorname{Int}\operatorname{Clo}\mathcal{C}$ be given, so that $\operatorname{Clo}\mathcal{C}$ includes a cell centered at y. If $q \neq y$, this cell, and hence $\operatorname{Clo}\mathcal{C}$, must have a non-empty intersection with $y + \mathbb{P}^{\times}(y-q)$. We choose z in this intersection, so that $z \in \operatorname{Clo}\mathcal{C}$ and $y \in \,]q,z[$. By Prop.3, we have $]q,z[\subset \operatorname{Int}\mathcal{C}$ and hence $y \in \operatorname{Int}\mathcal{C}$. Since $q \in \operatorname{Int}\mathcal{C}$, we see that $y \in \operatorname{Int}\mathcal{C}$ holds for all $y \in \operatorname{Int}\operatorname{Clo}\mathcal{C}$ and hence that $\operatorname{Int}\operatorname{Clo}\mathcal{C} \subset \operatorname{Int}\mathcal{C}$, which proves $(54.2)_2$. ∎

The following result states that the implication (53.6) can be reversed for convex sets.

Proposition 6: *If $\mathcal{C}$ is convex and not empty, then*

$$\operatorname{Int}\mathcal{C} \neq \emptyset \iff \operatorname{Fsp}\mathcal{C} = \mathcal{E}. \tag{54.3}$$

Proof: Assume that $\operatorname{Fsp}\mathcal{C} = \mathcal{E}$.

We choose $z \in \mathcal{C}$. We have $\operatorname{Lsp}(\mathcal{C}-z) = \mathcal{V}$ and hence we may choose a set basis $\mathfrak{b} \in \operatorname{Sub}(\mathcal{C}-z)$. Put $n := \dim\mathcal{E} = \dim\mathcal{V}$, and

$$q := z + \frac{1}{2n}\sum_{\mathbf{u}\in\mathfrak{b}} \mathbf{u}. \tag{54.4}$$

We claim that the cell $q + \frac{1}{2n}\operatorname{Box}\mathfrak{b}$ centered at q is included in $\mathcal{C}$, which implies that $q \in \operatorname{Int}\mathcal{C}$ and hence that $\operatorname{Int}\mathcal{C} \neq \emptyset$.

To prove the claim, let $x \in q + \frac{1}{2n}\operatorname{Box}\mathfrak{b}$ be given, so that $x = q + \frac{1}{2n}\sum(\lambda_{\mathbf{u}}\mathbf{u} \mid \mathbf{u}\in\mathfrak{b})$ for some $\lambda \in \,]-1,1[^{\mathfrak{b}}$. By (54.4), this means that

$$x = z + \sum_{\mathbf{u}\in\mathfrak{b}} \left(\frac{1+\lambda_{\mathbf{u}}}{2n}\right)\mathbf{u}.$$

Since $0 < \dfrac{1+\lambda_{\mathbf{u}}}{2} < 1$ for all $\mathbf{u}\in\mathfrak{b}$, and since $\sharp\mathfrak{b} = n$, we have

$$\sigma := \sum_{\mathbf{u}\in\mathfrak{b}} \frac{1+\lambda_{\mathbf{u}}}{2n} \in \,]0,1[.$$

It follows that x is the value of the convex combination mapping for $\{z\} \cup (z+\mathfrak{b})$ at the coefficient-family which assigns $(1-\sigma)$ to z and $\dfrac{1+\lambda_{\mathbf{u}}}{2n}$ to

$z + \mathbf{u}$ when $\mathbf{u} \in \mathfrak{b}$. Since $\mathcal{C}$ is convex and hence $\mathrm{Cxh}(\{z\} \cup (z + \mathfrak{b})) \subset \mathcal{C}$, it follows from Prop.5 of Sect.37 that $x \in \mathcal{C}$. Hence the claim is proved. ∎

Prop.6 shows that $(54.2)_2$ holds even when $\mathrm{Int}\,\mathcal{C} = \emptyset$, because in that case $\mathcal{C}$ and hence $\mathrm{Clo}\,\mathcal{C}$ must be included in a proper flat in $\mathcal{E}$. This flat and hence $\mathrm{Clo}\,\mathcal{C}$ has an empty interior.

55 Sequences

Let $\mathcal{E}$ be a flat space with translation space $\mathcal{V}$. Recall that the elements of $\mathcal{E}^{\mathbb{N}^\times}$ and $\mathcal{E}^{\mathbb{N}}$ are called *sequences* in $\mathcal{E}$ (see Sect.02). We will state most of our definitions and results for sequences indexed on $\mathbb{N}^\times$, but they are easily modified so as to apply to sequences indexed on $\mathbb{N}$. The following definitions are generalizations of ones that are familiar from real analysis (see Sect.08).

Definition 1: *We say that a sequence s in $\mathcal{E}$* **converges** *to a point $x \in \mathcal{E}$ if for every neighborhood $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ of x there is an $n \in \mathbb{N}^\times$ such that $s_>(n + \mathbb{N}) \subset \mathcal{N}$.*

Proposition 1: *A sequence in $\mathcal{E}$ can converge to at most one point in $\mathcal{E}$.*

Proof: Suppose that the sequence s converges to both x_1 and x_2 and that $x_1 \neq x_2$. choose a norm ν on $\mathcal{V}$. We then have $\sigma := \nu(x_1 - x_2) > 0$. Applying Def.1 to $\mathcal{N}_1 := x_1 + \frac{\sigma}{2}\mathrm{Ce}(\nu) \in \mathrm{Nhd}_{x_1}(\mathcal{E})$ and $\mathcal{N}_1 := x_2 + \frac{\sigma}{2}\mathrm{Ce}(\nu) \in \mathrm{Nhd}_{x_2}(\mathcal{E})$ we can determine $n_1, n_2 \in \mathbb{N}$ such that $s_>(n_1 + \mathbb{N}) \subset \mathcal{N}_1$ and $s_>(n_2 + \mathbb{N}) \subset \mathcal{N}_2$. Hence we have $s_{\max\{n_1,n_2\}} \in \mathcal{N}_1 \cap \mathcal{N}_2$, which contradicts the fact, easily verified, that $\mathcal{N}_1$ and $\mathcal{N}_2$ are disjoint. ∎

If the sequence s converges to x then x is called the **limit** of s and we write

$$x = \lim s = \lim_{n\to\infty} s_n$$

to express the assertion that the sequence $s := (s_n \mid n \in \mathbb{N}^\times)$ converges to x.

Definition 2: *We say that a point $x \in \mathcal{E}$ is a* **cluster point** *of a given sequence s in $\mathcal{E}$ or that s* **clusters** *at x if for every $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ and every $n \in \mathbb{N}^\times$ we have $s_>(n + \mathbb{N}) \cap \mathcal{N} \neq \emptyset$.*

If the sequence s converges to x, it is easily seen that x is the only cluster point of s.

Let ν be any norm on $\mathcal{V}$. In view of Prop.3 of Sect.53, one can replace the phrase "for every $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$" in both Def.1 and Def.2 by "for every ν-cell $\mathcal{N}$ centered at x." We conclude that

$$x = \lim\ s \iff \lim_{n\to\infty} \nu(s_n - x) = 0. \tag{55.1}$$

In particular, if $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ is a basis of $\mathcal{V}$ with dual $\mathbf{b}^*$ and if we apply (55.1) to the box-norm $\nu := \mathrm{no}_{\mathrm{Box}(\mathbf{b})}$ (see (51.16)), we obtain

$$x = \lim\ s \iff \lim_{n\to\infty} \mathbf{b}_i^*(s_n - x) = 0 \quad \text{for all} \quad i \in I. \tag{55.2}$$

We denote by $\mathrm{Conv}(\mathcal{E})$ the set of all convergent sequences in $\mathcal{E}$ and we define the mapping $\lim_{\mathcal{E}} : \mathrm{Conv}(\mathcal{E}) \to \mathcal{E}$ by $\lim_{\mathcal{E}}(s) = \lim\ s$. These notations apply, of course, if $\mathcal{E}$ is replaced by $\mathcal{V}$. Recall that the set $\mathcal{E}^{\mathbb{N}^\times}$ of all sequences in $\mathcal{E}$ has the natural structure of a flat space whose translation space is identified with the space $\mathcal{V}^{\mathbb{N}^\times}$ of all sequences in $\mathcal{V}$ (see Example 6 in Sect.33). The spaces $\mathcal{E}^{\mathbb{N}^\times}$ and $\mathcal{V}^{\mathbb{N}^\times}$ are infinite-dimensional except when $\mathcal{E}$ is a singleton. The action of $\mathcal{V}^{\mathbb{N}^\times}$ on $\mathcal{E}^{\mathbb{N}^\times}$ is given by $(s_n \mid n \in \mathbb{N}^\times) \mapsto (s_n + \mathbf{w}_n \mid n \in \mathbb{N}^\times)$ when $s \in \mathcal{E}^{\mathbb{N}^\times}$ and $\mathbf{w} \in \mathcal{V}^{\mathbb{N}^\times}$. The following two results are not hard to prove.

Proposition 2: $\mathrm{Conv}(\mathcal{E})$ *is a flat in* $\mathcal{E}^{\mathbb{N}^\times}$ *and* $\mathrm{Conv}(\mathcal{V})$ *is its direction space. The mapping* $\lim_{\mathcal{E}} : \mathrm{Conv}(\mathcal{E}) \to \mathcal{E}$ *is flat and its gradient is* $\lim_{\mathcal{V}} : \mathrm{Conv}(\mathcal{V}) \to \mathcal{V}$.

Proposition 3: *If* $\xi \in \mathrm{Conv}(\mathbb{R})$ *and* $\mathbf{w} \in \mathrm{Conv}(\mathcal{V})$ *then* $\xi\mathbf{w} := (\xi_n \mathbf{w}_n \mid n \in \mathbb{N}^\times)$ *belongs to* $\mathrm{Conv}(\mathcal{V})$ *and*

$$\lim\,(\xi\mathbf{w}) = (\lim\ \xi)(\lim\ \mathbf{w}). \tag{55.3}$$

Recall that a *subsequence* of a given sequence s is a sequence of the form $s\circ\sigma := (s_{\sigma(n)} \mid n \in \mathbb{N}^\times)$, where $\sigma : \mathbb{N}^\times \to \mathbb{N}^\times$ is strictly isotone (see Sect.08).

The proofs of the following two results are essentially the same as the proofs of the corresponding results in real analysis and will therefore be omitted.

Proposition 4: *Every subsequence of a given convergent sequence in* $\mathcal{E}$ *converges to the same point in* $\mathcal{E}$ *as the given sequence. In other words,* $x = \lim\ s$ *implies* $x = \lim(s \circ \sigma)$ *for all strictly isotone* $\sigma : \mathbb{N}^\times \to \mathbb{N}^\times$.

Proposition 5: *A point* x *is a cluster point of a given sequence if and only if a subsequence of the given sequence converges to* x.

We say that a sequence s in $\mathcal{E}$ is **bounded** if its range is a bounded subset of $\mathcal{E}$. The following theorem is a generalization of a basic theorem of real analysis (see Sect.08).

Cluster Point Theorem: *Every bounded sequence in a flat space has at least one cluster point.*

Proof: We proceed by induction over the dimension of the space. If this dimension is zero, then the assertion is trivial. Assume then, that $\mathcal{E}$ is a given flat space with $\dim\mathcal{E} > 0$ and that the assertion is valid for every hyperplane in $\mathcal{E}$. Choose a non-constant flat function $a \in \mathrm{Flf}(\mathcal{E})$. Since

$\nabla a \neq 0$, we may and do choose a $\mathbf{v} \in \mathcal{V}$ such that $(\nabla a)\mathbf{v} = 1$. We consider the hyperplane $\mathcal{F} := a^{<}(\{0\})$.

Now let s be a bounded sequence in $\mathcal{E}$. For each $n \in \mathbb{N}^{\times}$, we put

$$t_n := s_n - a(s_n)\mathbf{v}$$

so that $a(t_n) = a(s_n) - a(s_n)(\nabla a)\mathbf{v} = 0$, i.e. $t_n \in \mathcal{F}$. Therefore, $t := (t_n \mid n \in \mathbb{N}^{\times})$ is a sequence in $\mathcal{F}$, and it is easily seen that it is bounded in $\mathcal{F}$. By the induction hypothesis, t has a cluster point in $\mathcal{F}$ and hence, by Prop.5, there is a strictly isotone $\sigma : \mathbb{N}^{\times} \to \mathbb{N}^{\times}$ such that $t \circ \sigma$ converges. Now, the sequence $a \circ s \circ \sigma := (a(s_{\sigma(n)}) \mid n \in \mathbb{N}^{\times})$ is a real-valued bounded sequence. Therefore, by the Cluster Point Theorem of real analysis (see Sect.08) it has a convergent subsequence, i.e. there is a strictly isotone $\tau : \mathbb{N}^{\times} \to \mathbb{N}^{\times}$ such that the real-valued sequence $a \circ s \circ \sigma \circ \tau$ converges. By Prop.3, the sequence $(a \circ s \circ \sigma \circ \tau)\mathbf{v} := (a(s_{(\sigma\circ\tau)(n)})\mathbf{v} \mid n \in \mathbb{N}^{\times})$ converges in $\mathcal{V}$. By Prop.4, the subsequence $t \circ (\sigma \circ \tau)$ of the convergent sequence $t \circ \sigma$ converges in $\mathcal{F}$ and hence in $\mathcal{E}$. Therefore, by Prop.2, the sequence $s \circ (\sigma \circ \tau) = t \circ (\sigma \circ \tau) + (a \circ s \circ (\sigma \circ \tau))\mathbf{v}$ converges in $\mathcal{E}$. But this is a subsequence of s. In view of Prop.5, s has a cluster point. ■

The following result, a generalization of a basic fact of real analysis, enables one often to test a sequence for convergence without having a candidate for the purported limit.

Basic Convergence Criterion: *For every sequence s in $\mathcal{E}$ the following are equivalent:*

(i) *s converges.*

(ii) *For some norm ν on $\mathcal{V}$ and every $\varepsilon \in \mathbb{P}^{\times}$ there is an $m \in \mathbb{N}^{\times}$ such that*
$$\nu(s_{n+r} - s_n) < \varepsilon \quad \text{for all} \quad n \in m + \mathbb{N},\ r \in \mathbb{N}. \tag{55.4}$$

(iii) *For every $\mathcal{M} \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V})$ there is an $m \in \mathbb{N}^{\times}$ such that*
$$s_{>}(m + \mathbb{N}) - s_{>}(m + \mathbb{N}) \subset \mathcal{M}. \tag{55.5}$$

Proof: (i) $\Rightarrow$ (ii): Assume that the sequence s converges and put $x := \lim s$. Choose a norm ν on $\mathcal{V}$. Let $\varepsilon \in \mathbb{P}^{\times}$ be given. Using the equivalence (55.1) we can determine $m \in \mathbb{N}^{\times}$ such that $\nu(s_k - x) < \frac{\varepsilon}{2}$ for all $k \in m + \mathbb{N}$. Given $n \in m + \mathbb{N}$ and $r \in \mathbb{N}$, we have $n + r \in m + \mathbb{N}$ and hence

$$\nu(s_{n+r} - s_n) \leq \nu(s_{n+r} - x) + \nu(x - s_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves (55.4).

(ii) ⇒ (iii): Assume that (ii) holds and that $\mathcal{M} \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V})$ is given. By Prop.3 of Sect.53 we can determine $\varepsilon \in \mathbb{P}^\times$ such that $\varepsilon\,\mathrm{Ce}(\nu) \subset \mathcal{M}$. By (ii) we can determine $m \in \mathbb{N}$ such that (55.4) holds. Now, given $p, q \in s_>(m+\mathbb{N})$ we have either $p = s_n, q = s_{n+r}$ or $q = s_n, p = s_{n+r}$ for suitable $n \in m + \mathbb{N}$, $r \in \mathbb{N}$ and hence $p-q = \pm(s_{n+r}-s_n)$. It follows from (55.4) that $\nu(p-q) < \varepsilon$ and hence $p - q \in \varepsilon\,\mathrm{Ce}(\nu) \subset \mathcal{M}$. Since $p, q \in s_>(m + \mathbb{N})$ were arbitrary, we conclude that (55.5) holds.

(iii) ⇒ (i): Assume that (iii) holds. We choose a norming cell $\mathcal{B}_o$ and determine $m \in \mathbb{N}$ such that (55.5) holds with $\mathcal{M}$ replaced by $\mathcal{B}_o$. We then have $s_>(m + \mathbb{N}) - s_m \subset \mathcal{B}_o$ and hence

$$\mathrm{Rng}\, s = s_>((m-1)^{]}) \cup s_>(m+\mathbb{N}) \;\subset\; s_>((m-1)^{]}) \cup (s_m + \mathcal{B}_o).$$

Since $s_m + \mathcal{B}_o$ is bounded by Prop.3 of Sect.52 and since $s_>((m-1)^{]})$ is finite and hence bounded, it follows that $\mathrm{Rng}\, s$ is bounded. Therefore, by the Cluster Point Theorem we may and do choose a cluster point x of s. Now let $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ be given. We choose a norming cell $\mathcal{B}$ included in $\mathcal{N} - x$ and determine $n \in \mathbb{N}^\times$ such that (55.5) holds with $\mathcal{M}$ replaced by $\frac{1}{2}\mathcal{B}$, so that $s_>(n+\mathbb{N}) - s_>(n+\mathbb{N}) \subset \frac{1}{2}\mathcal{B}$. By Def.2, we can choose $z \in s_>(n+\mathbb{N}) \cap (x + \frac{1}{2}\mathcal{B})$. We then have $s_>(n + \mathbb{N}) - z \;\subset\; s_>(n+\mathbb{N}) - s_>(n+\mathbb{N}) \;\subset\; \frac{1}{2}\mathcal{B}$ and hence

$$s_>(n+\mathbb{N}) \;\subset\; z + \frac{1}{2}\mathcal{B} \;\subset\; (x + \frac{1}{2}\mathcal{B}) + \frac{1}{2}\mathcal{B} = x + \mathcal{B} \subset \mathcal{N}.$$

Since $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ was arbitrary it follows that s converges to x.

Proposition 6: *Let $\mathcal{S}$ be a subset of $\mathcal{E}$ and let $x \in \mathcal{E}$. Then $x \in \mathrm{Clo}\,\mathcal{S}$ if and only if x is the limit [a cluster point] of some sequence in $\mathcal{S}$.*

Proof: Assume that $x \in \mathrm{Clo}\,\mathcal{S}$. We choose a norming cell $\mathcal{B}$. Since, for each $n \in \mathbb{N}^\times$, $x + \frac{1}{n}\mathcal{B} \in \mathrm{Nhd}_x(\mathcal{E})$, it follows from (53.3) that $(x+\frac{1}{n}\mathcal{B}) \cap \mathcal{S} \neq \emptyset$ for all $n \in \mathbb{N}^\times$. Hence we may choose a sequence $s := (s_n \mid n \in \mathbb{N}^\times)$ in $\mathcal{S}$ such that $s_n \in x + \frac{1}{n}\mathcal{B}$. It is clear that x is the limit (and hence a cluster point) of s.

Assume now that $s := (s_n \mid n \in \mathbb{N}^\times)$ is a sequence in $\mathcal{S}$ and that x is a cluster point (or even a limit) of s. Let $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ be given. In view of Def.2, there is an $n \in \mathbb{N}^\times$ such that $s_n \in \mathcal{N}$. Since $s_n \in \mathcal{S}$ it follows that $\mathcal{S} \cap \mathcal{N} \neq \emptyset$. Since $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ was arbitrary, it follows from (53.3) that $x \in \mathrm{Clo}\,\mathcal{S}$. ∎

We now assume that flat spaces $\mathcal{E}, \mathcal{E}'$, with translation spaces $\mathcal{V}, \mathcal{V}'$, and subsets $\mathcal{D} \subset \mathcal{E}$, $\mathcal{D}' \subset \mathcal{E}'$ are given. We consider sequences of mappings from $\mathcal{D}$ to $\mathcal{D}'$, i.e. elements of $(\mathrm{Map}\,(\mathcal{D}, \mathcal{D}'))^{\mathbb{N}^\times}$.

Definition 3: *We say that a sequence* $\sigma = (\sigma_n \mid n \in \mathbb{N}^\times)$ *in* $\mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ **converges** *to* $\varphi \in \mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ *if*

$$\varphi(x) = \lim_{n\to\infty} \sigma_n(x) \quad \text{for all} \quad x \in \mathcal{D}.$$

We say that σ *converges* **uniformly** *to* φ *if for every* $\mathcal{M}' \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V}')$ *there is an* $n \in \mathbb{N}^\times$ *such that*

$$\sigma_m(x) \in \varphi(x) + \mathcal{M}' \quad \text{for all} \quad m \in n + \mathbb{N},\ x \in \mathcal{D}. \tag{55.6}$$

We say that σ *converges* **locally uniformly** *to* φ *if for every* $x \in \mathcal{D}$ *there is* $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ *such that the sequence* $(\sigma_n|_{\mathcal{N}} \mid n \in \mathbb{N}^\times)$ *in* $\mathrm{Map}\,(\mathcal{N} \cap \mathcal{D}, \mathcal{D}')$ *converges uniformly to* $\varphi|_{\mathcal{N}} \in \mathrm{Map}\,(\mathcal{N} \cap \mathcal{D}, \mathcal{D}')$.

It is evident that uniform convergence implies locally uniform convergence, and that the latter implies convergence.

We write

$$\varphi = \lim \sigma = \lim_{n\to\infty} \sigma_n$$

to express the assertion that the sequence σ converges to φ.

Proposition 7: *Let* ν' *be a norm on* $\mathcal{V}'$. *Then the sequence* σ *in* $\mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ *converges uniformly to* $\varphi \in \mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ *if and only if for every* $\varepsilon \in \mathbb{P}^\times$ *there is* $m \in \mathbb{N}^\times$ *such that*

$$\nu' \circ (\sigma_n - \varphi) < \varepsilon \quad \text{for all} \quad n \in m + \mathbb{N} \tag{55.7}$$

where $\sigma_n - \varphi \in \mathrm{Map}\,(\mathcal{D}, \mathcal{V}')$ *is the value-wise difference of* σ_n *and* φ.

Proof: If $\mathcal{M}' := \varepsilon\mathrm{Ce}(\nu')$, the defining condition (55.6) reduces to (55.7). Hence the assertion is a consequence of Prop.3 of Sect.53. ■

The following criterion for uniform convergence is an analogue of the Basic Convergence Criterion.

Proposition 8: *Assume that* $\mathcal{D}'$ *is a closed subset of* $\mathcal{E}'$. *Let* ν' *be a norm on* $\mathcal{V}'$. *A sequence* σ *in* $\mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ *converges uniformly to some mapping in* $\mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ *if and only if, for every* $\varepsilon \in \mathbb{P}^\times$, *there is* $m \in \mathbb{N}^\times$ *such that*

$$\nu' \circ (\sigma_{n+r} - \sigma_n) < \varepsilon \quad \text{for all} \quad n \in m + \mathbb{N},\ r \in \mathbb{N}. \tag{55.8}$$

Proof: Assume that σ converges uniformly to φ. Let $\varepsilon \in \mathbb{P}^\times$ be given. By Prop.7, we can determine $m \in \mathbb{N}^\times$ such that $\nu' \circ (\sigma_k - \varphi) < \frac{\varepsilon}{2}$ for all $k \in m + \mathbb{N}$. Given $n \in m + \mathbb{N}$ and $r \in \mathbb{N}$, we have $n + r \in m + \mathbb{N}$ and hence

$$\nu' \circ (\sigma_{n+r} - \sigma_n) \le \nu' \circ (\sigma_{n+r} - \varphi) + \nu' \circ (\varphi - \sigma_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves (55.8).

Assume now that σ satisfies the condition. Then, for each $x \in \mathcal{D}$, the sequence $(\sigma_n(x) \mid n \in \mathbb{N}^\times)$ satisfies the condition (ii) of the Basic Convergence Criterion and hence converges. By Prop.6, its limit belongs to $\mathcal{D}'$ because $\mathcal{D}'$ is closed. Hence we can define $\varphi : \mathcal{D} \to \mathcal{D}'$ by

$$\varphi(x) := \lim_{n\to\infty} \sigma_n(x) \quad \text{for all} \quad x \in \mathcal{D}. \tag{55.9}$$

Let $\varepsilon \in \mathbb{P}^\times$ be given. Determine $m \in \mathbb{N}^\times$ such that (55.8) holds with ε replaced by $\frac{\varepsilon}{2}$. Given $n \in m + \mathbb{N}^\times$ and $x \in \mathcal{D}$, we then have

$$\nu'(\sigma_{n+r}(x) - \sigma_n(x)) < \frac{\varepsilon}{2} \quad \text{for all} \quad r \in \mathbb{N}.$$

By (55.9) we can determine $r \in \mathbb{N}$ such that $\nu'(\varphi(x) - \sigma_{n+r}(x)) < \frac{\varepsilon}{2}$. We conclude that

$$\nu'(\varphi(x) - \sigma_n(x)) \le \nu'(\varphi(x) - \sigma_{n+r}(x)) + \nu'(\sigma_{n+r}(x) - \sigma_n(x)) < \varepsilon.$$

Since $n \in m + \mathbb{N}^\times$ and $x \in \mathcal{D}$ were arbitrary, it follows that (55.7) holds and hence, by Prop.7, that σ converges uniformly. ■

Given a sequence $\mathbf{w}$ in $\mathrm{Map}\,(\mathcal{D}, \mathcal{V}')$ and indexed on $\mathbb{N}$ we define the **sum-sequence** $\mathrm{ssq}\,\mathbf{w}$ of $\mathbf{w}$ by

$$(\mathrm{ssq}\,\mathbf{w})_n := \sum_{k\in n^[} \mathbf{w}_k \quad \text{for all} \quad n \in \mathbb{N} \tag{55.10}$$

(compare with (08.22)); it is again a sequence in $\mathrm{Map}\,(\mathcal{D}, \mathcal{V}')$ and indexed on $\mathbb{N}$. The following result is an easy consequence of Prop.8.

Proposition 9: *Let* $\mathbf{w}$ *be a sequence in* $\mathrm{Map}\,(\mathcal{D}, \mathcal{V}')$. *Let* ν' *be a norm on* $\mathcal{V}'$ *and let* a *be a sequence in* $\mathbb{P}$ *such that* $\nu' \circ \mathbf{w}_n \le a_n$ *for all* $n \in \mathbb{N}$ *and such that the sum-sequence* $\mathrm{ssq}\ a \in \mathbb{P}^{\mathbb{N}}$ *of* a *converges. Then the sum-sequence* $\mathrm{ssq}\,\mathbf{w}$ *of* $\mathbf{w}$ *converges uniformly to a mapping in* $\mathrm{Map}\,(\mathcal{D}, \mathcal{V}')$.

Proof: Let $\varepsilon \in \mathbb{P}^\times$ be given. By condition (ii) of the Basic Convergence Criterion, applied to the sum-sequence of a, we can determine $m \in \mathbb{N}$ such that

$$\sum_{k\in(n+r)^[\setminus n^[} a_k = (\mathrm{ssq}\ a)_{n+r} - (\mathrm{ssq}\ a)_n < \varepsilon$$

for all $n \in m + \mathbb{N}$ and all $r \in \mathbb{N}$. By (51.13) we have

$$\begin{aligned} \nu' \circ ((\operatorname{ssq} \mathbf{w})_{n+r} - (\operatorname{ssq}\ \mathbf{w})_n) &= \nu' \circ (\sum_{k \in (n+r)^{[} \setminus n^{[}} \mathbf{w}_k) \\ &\leq \sum_{k \in (n+r)^{[} \setminus n^{[}} (\nu' \circ \mathbf{w}_k) \\ &\leq \sum_{k \in (n+r)^{[} \setminus n^{[}} a_k < \varepsilon \end{aligned}$$

for all $n \in m + \mathbb{N}$ and all $r \in \mathbb{N}$. Hence (55.8) holds with $\sigma := \operatorname{ssq} \mathbf{w}$ and Prop.8 gives the desired conclusion. ∎

Notes 55

(1) The Notes (4), (5), and (6) to Sect.08 also apply to this section.

(2) The Basic Convergence Criterion is often called the "Cauchy Convergence Criterion". A sequence that satisfies this criterion is often called a "Cauchy sequence" or a "fundamental sequence".

(3) The test for uniform convergence of a sum-sequence implied by Prop.9 is often called the "Weierstrass Comparison Test" or "Weierstrass M-Test".

56 Continuity, Uniform Continuity

Let $\mathcal{E}$ be a flat space, $\mathcal{D}$ a subset of $\mathcal{E}$, and $x \in \mathcal{E}$. We use the notation

$$\operatorname{Nhd}_x(\mathcal{D}) := \{\mathcal{N} \cap \mathcal{D} \mid \mathcal{N} \in \operatorname{Nhd}_x(\mathcal{E})\}. \tag{56.1}$$

In the special case when $\mathcal{D}$ happens to be a flat, we see from (53.16) that the notation (56.1) is consistent with the notation for the collection of all neighborhoods of x in the flat space $\mathcal{D}$. We call the members of $\operatorname{Nhd}_x(\mathcal{D})$ **neighborhoods relative to** $\mathcal{D}$ of x.

We say that a subset $\mathcal{H}$ of $\mathcal{D}$ is **open relative to** $\mathcal{D}$ if $\mathcal{H}$ is a neighborhood relative to $\mathcal{D}$ of each point of $\mathcal{H}$. If this is the case, we have $\mathcal{H} = \mathcal{D} \cap \mathcal{H}^\dagger$, where

$$\mathcal{H}^\dagger := \bigcup \{\mathcal{S} \in \operatorname{Sub} \mathcal{E} \mid \mathcal{S} \text{ is open}, \quad \mathcal{D} \cap \mathcal{S} \subset \mathcal{H}\} \tag{56.2}$$

is an open subset of $\mathcal{E}$. Conversely, if $\mathcal{H} = \mathcal{D} \cap \mathcal{G}$ for some open subset $\mathcal{G}$ of $\mathcal{E}$, then $\mathcal{H}$ is clearly open relative to $\mathcal{D}$.

If $\mathcal{D}$ is an open subset of $\mathcal{E}$, then

$$\mathrm{Nhd}_x(\mathcal{D}) = \mathrm{Nhd}_x(\mathcal{E}) \cap \mathrm{Sub}\,\mathcal{D}.$$

Moreover, the subsets of $\mathcal{D}$ that are open relative to $\mathcal{D}$ are simply the open subsets of $\mathcal{D}$.

We now assume that flat spaces $\mathcal{E}, \mathcal{E}'$ with translation spaces $\mathcal{V}, \mathcal{V}'$ and subsets $\mathcal{D} \subset \mathcal{E}$, $\mathcal{D}' \subset \mathcal{E}'$ are given.

Definition 1: *We say that a given mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ is* **continuous at** *a given $x \in \mathcal{D}$ if the pre-image under φ of every neighborhood of $\varphi(x)$ relative to $\mathcal{D}'$ is a neighborhood of x relative to $\mathcal{D}$, i.e. if*

$$(\varphi^{<})_{>}(\mathrm{Nhd}_{\varphi(x)}(\mathcal{D}')) \subset \mathrm{Nhd}_x(\mathcal{D}). \tag{56.3}$$

We say that $\varphi : \mathcal{D} \to \mathcal{D}'$ is **continuous** *if it is continuous at every point $x \in \mathcal{D}$.*

Using Prop.3 of Sect.53 we immediately obtain the following criterion.

Proposition 1: *Let norms ν and ν' on $\mathcal{V}$ and $\mathcal{V}'$ be given. The mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ is continuous at $x \in \mathcal{D}$ if and only if for every $\varepsilon \in \mathbb{P}^\times$ there is a $\delta \in \mathbb{P}^\times$ such that*

$$\big(\nu(y - x) < \delta \implies \nu'(\varphi(y) - \varphi(x)) < \varepsilon\big) \quad \text{for all} \quad y \in \mathcal{D}. \tag{56.4}$$

Proposition 2: *The mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ is continuous at $x \in \mathcal{D}$ if and only if for every sequence $s := (s_n \mid n \in \mathbb{N}^\times)$ in $\mathcal{D}$ that converges to x, the sequence $\varphi \circ s = (\varphi(s_n) \mid n \in \mathbb{N}^\times)$ converges to $\varphi(x)$, i.e.*

$$\lim\,(\varphi \circ s) = \varphi(\lim\ s). \tag{56.5}$$

Proof: Assume that φ is continuous at x. Let a sequence s in $\mathcal{D}$ that converges to x and $\mathcal{N}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{E}')$ be given. Then $\mathcal{N}' \cap \mathcal{D}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{D}')$ and hence, since φ is continuous at x, $\varphi^{<}(\mathcal{N}' \cap \mathcal{D}') \in \mathrm{Nhd}_x(\mathcal{D})$. Hence we may choose $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ such that $\varphi_{>}(\mathcal{N} \cap \mathcal{D}) \subset \mathcal{N}' \cap \mathcal{D}' \subset \mathcal{N}'$. Since s converges to x and since $\mathrm{Rng}\, s \subset \mathcal{D}$ there is $n \in \mathbb{N}^\times$ such that $s_{>}(n + \mathbb{N}) \subset \mathcal{N} \cap \mathcal{D}$. It follows that

$$(\varphi \circ s)_{>}(n + \mathbb{N}) = \varphi_{>}(s_{>}(n + \mathbb{N})) \subset \mathcal{N}'.$$

Since $\mathcal{N}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{E}')$ was arbitrary, it follows that $\varphi \circ s$ converges to $\varphi(x)$.

Assume now that φ fails to be continuous at x. We choose a norming cell $\mathcal{B}$ in $\mathcal{V}$ and a neighborhood $\mathcal{N}'$ of $\varphi(x)$ in $\mathcal{E}'$ such that $\varphi^{<}(\mathcal{N}' \cap \mathcal{D}')$ is not a neighborhood of x relative to $\mathcal{D}$. Since, for every $n \in \mathbb{N}^\times$, we have $\mathcal{D} \cap (x + \frac{1}{n}\mathcal{B}) \in \mathrm{Nhd}_x(\mathcal{D})$, it follows that $\varphi_>(\mathcal{D} \cap (x + \frac{1}{n}\mathcal{B})) \not\subset \mathcal{N}'$ for every $n \in \mathbb{N}^\times$. Hence we may choose a sequence $s := (s_n \mid n \in \mathbb{N}^\times)$ in $\mathcal{D}$ such that $s_n \in x + \frac{1}{n}\mathcal{B}$ but $(\varphi \circ s)_n = \varphi(s_n) \notin \mathcal{N}'$ for all $n \in \mathbb{N}^\times$. It is clear that $x = \lim s$ but that $\varphi \circ s$ cannot converge to $\varphi(x)$. ∎

Proposition 3: *Assume that $\mathcal{D}$ and $\mathcal{D}'$ are subsets of $\mathcal{E}$ and $\mathcal{E}'$, respectively. Then $\varphi : \mathcal{D} \to \mathcal{D}'$ is continuous if and only if the pre-image under φ of every subset of $\mathcal{D}'$ that is open relative to $\mathcal{D}'$ is a subset of $\mathcal{D}$ that is open relative to $\mathcal{D}$.*

Proof: Assume that φ is continuous. If $\mathcal{H}'$ is a subset of $\mathcal{D}'$ that is open relative to $\mathcal{D}'$ then $\mathcal{H}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{D}')$ for every $x \in \varphi^<(\mathcal{H}')$ and hence, by (56.3) $\varphi^<(\mathcal{H}') \in \mathrm{Nhd}_x(\mathcal{D})$ for every $x \in \varphi^<(\mathcal{H}')$. It follows that $\varphi^<(\mathcal{H}')$ is open relative to $\mathcal{D}$.

Assume now that the condition is satisfied. Let $x \in \mathcal{D}$ and $\mathcal{N}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{D}')$ be given. Using (56.1) we see that $\mathcal{N}' = \mathcal{M}' \cap \mathcal{D}'$ for some $\mathcal{M}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{E}')$. We then have $\varphi(x) \in \mathrm{Int}\,(\mathcal{M}')$, and, since $\mathrm{Int}\,(\mathcal{M}')$ is an open subset of $\mathcal{E}'$, it follows that $\mathrm{Int}\,(\mathcal{M}') \cap \mathcal{D}'$ is open relative to $\mathcal{D}'$. Therefore $\varphi^<(\mathrm{Int}\,(\mathcal{M}') \cap \mathcal{D}')$ is open relative to $\mathcal{D}$ and hence a neighborhood relative to $\mathcal{D}$ of x. Since $\varphi^<(\mathrm{Int}\,(\mathcal{M}') \cap \mathcal{D}) \subset \varphi^<(\mathcal{M}' \cap \mathcal{D}) = \varphi^<(\mathcal{N}')$, it follows that $\varphi^<(\mathcal{N}') \in \mathrm{Nhd}_x(\mathcal{D})$. Therefore, since $x \in \mathcal{D}$ and $\mathcal{N}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{D}')$ were arbitrary, φ is continuous. ∎

Definition 2: *We say that a given mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ is* **uniformly continuous** *if for every $\mathcal{M}' \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V}')$ there is a $\mathcal{M} \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V})$ such that for all $x, y \in \mathcal{D}$,*

$$y - x \in \mathcal{M} \implies \varphi(y) - \varphi(x) \in \mathcal{M}'. \tag{56.6}$$

Using Prop.3 of Sect.53 we obtain the following analogue of Prop.1:

Proposition 4: *Let norms ν and ν' on $\mathcal{V}$ and $\mathcal{V}'$ be given. The mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ is uniformly continuous if and only if for every $\varepsilon \in \mathbb{P}^\times$ there is a $\delta \in \mathbb{P}^\times$ such that*

$$\big(\nu(y - x) < \delta \implies \nu'(\varphi(y) - \varphi(x)) < \varepsilon\big) \quad \text{for all} \quad x, y \in \mathcal{D}. \tag{56.7}$$

Note that the criterion for uniform continuity of Prop.4 differs from the criterion for continuity obtained from Prop.1 only by the placement of the quantifier "for all x", and hence that uniform continuity implies continuity.

Proposition 5: *Every flat mapping is uniformly continuous.*

Proof: Let $\alpha : \mathcal{E} \to \mathcal{E}'$ be a flat mapping, so that

$$\alpha(y) - \alpha(x) = \nabla\alpha(y - x) \quad \text{for all} \quad x, y \in \mathcal{E}.$$

We choose norms ν and ν' on $\mathcal{V}$ and $\mathcal{V}'$. By (52.7) we have

$$\nu'(\alpha(y) - \alpha(x)) = \nu'(\nabla\alpha(y - x)) \leq ||\nabla\alpha||_{\nu,\nu'}\nu(y - x)$$

for all $x, y \in \mathcal{E}$. It follows that (56.7) holds when $\delta \in \mathbb{P}^\times$ is chosen such that $||\nabla\alpha||_{\nu,\nu'}\delta < \varepsilon$. ∎

Composition Theorem for Continuity: *Let $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}''$ be subsets of flat spaces. If $\varphi : \mathcal{D} \to \mathcal{D}'$ is continuous at $x \in \mathcal{D}$ and if $\psi : \mathcal{D}' \to \mathcal{D}''$ is continuous at $\varphi(x) \in \mathcal{D}'$, then $\psi \circ \varphi : \mathcal{D} \to \mathcal{D}''$ is continuous at x.*

The composite of two continuous mappings is again continuous.

Proof: Since φ is continuous at x, (56.3) is valid. Since ψ is continuous at $\varphi(x)$, we also have

$$(\psi^<)_>(\mathrm{Nhd}_{\psi(\varphi(x))}(\mathcal{D}'')) \subset \mathrm{Nhd}_{\varphi(x)}(\mathcal{D}')$$

and hence

$$(\varphi^<)_>((\psi^<)_>(\mathrm{Nhd}_{(\psi\circ\varphi)(x)}(\mathcal{D}''))) \subset (\varphi^<)_>(\mathrm{Nhd}_{\varphi(x)}(\mathcal{D})).$$

Since $((\psi \circ \varphi)^<)_> = (\varphi^< \circ \psi^<)_> = (\varphi^<)_> \circ (\psi^<)_>$ (see (03.15)), we obtain, using (56.3), the inclusion

$$((\psi \circ \varphi)^<)_>(\mathrm{Nhd}_{(\psi\circ\varphi)(x)}(\mathcal{D}'') \subset \mathrm{Nhd}_x(\mathcal{D}),$$

which means that $\psi \circ \varphi$ is continuous at x. ∎

Composition Theorem for Uniform Continuity: *The composite of two uniformly continuous mappings is again uniformly continuous.*

Proof: Let $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ be flat spaces with translation spaces $\mathcal{V}, \mathcal{V}', \mathcal{V}''$ and $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be subsets of $\mathcal{E}, \mathcal{E}', \mathcal{E}''$, respectively. Assume that $\varphi : \mathcal{D} \to \mathcal{D}'$ and $\psi : \mathcal{D}' \to \mathcal{D}''$ are uniformly continuous. Let $\mathcal{M}'' \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V}'')$ be given. The uniform continuity of ψ implies that we can choose $\mathcal{M}' \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V}')$ such that

$$y' - x' \in \mathcal{M}' \implies \psi(y') - \psi(x') \in \mathcal{M}'' \tag{56.8}$$

for all $x', y' \in \mathcal{D}'$. Since φ is uniformly continuous, we can find $\mathcal{M} \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V})$ such that

$$y - x \in \mathcal{M} \implies \varphi(x) - \varphi(y) \in \mathcal{M}' \tag{56.9}$$

for all $x, y \in \mathcal{D}$. Using (56.8) with the choice $y' := \varphi(y),\ x' := \varphi(x)$ and combining the result with (56.9), we obtain

$$y - x \in \mathcal{M} \implies (\psi \circ \varphi)(y) - (\psi \circ \varphi)(x) \in \mathcal{M}''.$$

Since $\mathcal{M}'' \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V}'')$ was arbitrary, we obtain the asserted uniform continuity of $\psi \circ \varphi$. ∎

Proposition 6: *Let $\mathcal{E}$ be a flat space with translation space $\mathcal{V}$, let $\mathcal{S}$ be a non-empty subset of $\mathcal{E}$ and let ν be a norm on $\mathcal{V}$. Then the function $d : \mathcal{E} \to \mathbb{P}$ defined by*

$$d(x) := \inf\{\nu(x - z) \mid z \in \mathcal{S}\} \tag{56.10}$$

is uniformly continuous. Moreover, $\mathrm{Clo}\,\mathcal{S} = d^{<}(\{0\})$, *and, for each $\delta \in \mathbb{P}^{\times}$, we have*

$$\mathcal{S} + \delta\overline{\mathrm{Ce}}(\nu) \subset d^{<}([0, \delta]). \tag{56.11}$$

Proof: Let $x, y \in \mathcal{E}$ be given. By (56.10) and (N_2) of Def.3 of Sect.51, we have

$$d(x) \leq \nu(x - z) \leq \nu(x - y) + \nu(y - z), \quad \text{for all} \quad z \in \mathcal{S}.$$

Hence, using (56.10) again, we obtain

$$d(x) \leq \nu(x - y) + d(y).$$

This inequality and the one obtained from it by interchanging the roles of x and y give

$$|d(x) - d(y)| \leq \nu(x - y).$$

Since $x, y \in \mathcal{E}$ were arbitrary, the asserted uniform continuity of d follows from Prop.4.

Let $x \in \mathcal{E}$ and $\delta \in \mathbb{P}^{\times}$ be given. Then the following are evidently equivalent:

(i) $(x + \delta\overline{\mathrm{Ce}}(\nu)) \cap \mathcal{S} \neq \emptyset$,

(ii) $q \in x + \delta\overline{\mathrm{Ce}}(\nu)$ for some $q \in \mathcal{S}$,

(iii) $x \in \mathcal{S} + \delta\overline{\mathrm{Ce}}(\nu)$,

(iv) $\nu(q - x) \leq \delta$ and some $q \in \mathcal{S}$.

In view of (53.3) and Prop.3 of Sect.53, the equivalence (i) $\Leftrightarrow$ (iv) shows that $x \in \mathrm{Clo}\,\mathcal{S}$ if and only if for every $\delta \in \mathbb{P}^{\times}$ there is a $q \in \mathcal{S}$ such that $\nu(q - x) \leq \delta$. In view of (56.10), this condition is equivalent to $d(x) = 0$ and hence we have $\mathrm{Clo}\,\mathcal{S} = d^{<}(\{0\})$.

If (iv) holds then $d(x) \leq \delta$. Hence (iii) implies $d(x) \leq \delta$, which shows that the inclusion (56.11) holds. ∎

In the case when $\mathcal{E} := \mathcal{V}$ and $\mathcal{S} := \{\mathbf{0}\}$, the definition (56.10) gives $d = \nu$ and Prop.6 yields

Proposition 7: *Every norm on a linear space is uniformly continuous.*

The value $d(x)$ given by (56.10) is called the **distance** of the point x from the set $\mathcal{S}$, relative to the norm ν. If $\mathcal{E}$ is a genuine Euclidean space, if ν is the magnitude $|\cdot|$, and if $\mathcal{S} := \{y\}$, then $d(x) = \mathrm{dst}(x, y)$, where dst is the distance function of Sect.46.

We now assume that flat spaces $\mathcal{E}, \mathcal{E}'$, with translation spaces $\mathcal{V}, \mathcal{V}'$, and subsets $\mathcal{D} \subset \mathcal{E}$ and $\mathcal{D}' \subset \mathcal{E}'$ are given.

Theorem on Continuity of Uniform Limits: *Let σ be a sequence of continuous mappings in* $\mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ *that converges locally uniformly to* $\varphi \in \mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$. *Then φ is continuous.*

Proof: Let $x \in \mathcal{D}$ and $\mathcal{N}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{E}')$ be given. We choose a norming cell $\mathcal{B}'$ in $\mathcal{V}'$ such that $\varphi(x) + 3\mathcal{B}' \subset \mathcal{N}'$. In view of Def.3 of Sect.55, we can determine $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ and $n \in \mathbb{N}^\times$ such that

$$\sigma_n(y) - \varphi(y) \in \mathcal{B}' \quad \text{for all} \quad y \in \mathcal{N} \cap \mathcal{D}. \tag{56.12}$$

Since σ_n is continuous at x, in view of Def.1 we have $\mathcal{M} := \sigma_n^{<}(\sigma_n(x) + \mathcal{B}') \in \mathrm{Nhd}_x(\mathcal{D})$, so that

$$\sigma_n(y) - \sigma_n(x) \in \mathcal{B}' \quad \text{for all} \quad y \in \mathcal{M}. \tag{56.13}$$

Using (56.12) twice and (56.13) we obtain

$$\begin{array}{rcl} \varphi(y) - \varphi(x) & = & (\varphi(y) - \sigma_n(y)) + (\sigma_n(y) - \sigma_n(x)) + (\sigma_n(x) - \varphi(x)) \\ & \in & (-\mathcal{B}') + \mathcal{B}' + \mathcal{B}' = 3\mathcal{B}' \end{array}$$

for all $y \in \mathcal{N} \cap \mathcal{D} \cap \mathcal{M}$, i.e. $\varphi(y) \in \varphi(x) + 3\mathcal{B}' \subset \mathcal{N}'$ for all $y \in \mathcal{N} \cap \mathcal{D} \cap \mathcal{M}$. Since $\mathcal{N} \cap \mathcal{D} \cap \mathcal{M} \in \mathrm{Nhd}_x(\mathcal{D})$, this proves the continuity of φ at x. ∎

57 Limits

Definition 1: *Let $\mathcal{D}$ be a subset of a flat space $\mathcal{E}$. We say that $x \in \mathcal{E}$ is an* **accumulation point** *of $\mathcal{D}$ if $x \in \mathrm{Clo}\,(\mathcal{D} \setminus \{x\})$. We denote the set of all accumulation points of $\mathcal{D}$ by* $\mathrm{Acc}\mathcal{D}$.

We have $\mathrm{Clo}\,\mathcal{D} = \mathcal{D} \cup \mathrm{Acc}\mathcal{D}$, i.e. every point in $\mathrm{Clo}\,\mathcal{D}$ either belongs to $\mathcal{D}$ or is an accumulation point of $\mathcal{D}$ (or both). The points of $\mathcal{D}$ that are not also accumulation points of $\mathcal{D}$ are called **isolated points** of $\mathcal{D}$.

Proposition 1: *Let $\mathcal{D}, \mathcal{D}'$ be subsets of flat spaces $\mathcal{E}, \mathcal{E}'$. Given $\varphi : \mathcal{D} \to \mathcal{D}'$ and $x \in \mathrm{Acc}\mathcal{D}$, there is at most one $x' \in \mathcal{E}'$ such that the mapping $\overline{\varphi} : \mathcal{D} \cup \{x\} \to \mathcal{D}' \cup \{x'\}$ defined by*

$$\overline{\varphi}(z) := \left\{ \begin{array}{ccc} \varphi(z) & \text{if} & z \in \mathcal{D} \setminus \{x\} \\ x' & \text{if} & z = x \end{array} \right\} \tag{57.1}$$

is continuous at x.

Proof: Suppose that $x_1' \in \mathcal{E}'$ and $x_2' \in \mathcal{E}'$, with $x_1' \neq x_2'$, both satisfy the condition and that $\overline{\varphi}_1 : \mathcal{D} \cup \{x\} \to \mathcal{D}' \cup \{x_1'\}$, $\overline{\varphi}_2 : \mathcal{D} \cup \{x\} \to \mathcal{D}' \cup \{x_2'\}$ are defined according to (57.1). Choose a norm ν' on $\mathcal{V}'$. We then have $\sigma := \nu'(x_1' - x_2') > 0$ and hence $\mathcal{N}_1' \cap \mathcal{N}_2' = \emptyset$ when $\mathcal{N}_1' := x_1' + \frac{\sigma}{2}\mathrm{Ce}(\nu') \in \mathrm{Nhd}_{x_1'}(\mathcal{E}')$ and $\mathcal{N}_2' := x_2' + \frac{\sigma}{2}\mathrm{Ce}(\nu') \in \mathrm{Nhd}_{x_2'}(\mathcal{E}')$. Since $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are both continuous at x, in view of Def.1 of Sect.56 and (56.1) we can determine $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{Nhd}_x(\mathcal{E})$ such that

$$\overline{\varphi_1}^{<}(\mathcal{N}_1') = \mathcal{N}_1 \cap (\mathcal{D} \cup \{x\}), \quad \overline{\varphi_2}^{<}(\mathcal{N}_2') = \mathcal{N}_2 \cap (\mathcal{D} \cup \{x\}).$$

Since, by (57.1), both $\overline{\varphi_1}$ and $\overline{\varphi_2}$ agree with φ on $\mathcal{D} \setminus \{x\}$, we conclude that

$$\varphi_{>}(\mathcal{N}_1 \cap \mathcal{N}_2 \cap (\mathcal{D} \setminus \{x\})) \subset \mathcal{N}_1' \cap \mathcal{N}_2' = \emptyset$$

and hence $(\mathcal{N}_1 \cap \mathcal{N}_2) \cap (\mathcal{D} \setminus \{x\}) = \emptyset$. Since $\mathcal{N}_1 \cap \mathcal{N}_2 \in \mathrm{Nhd}_x(\mathcal{E})$, it follows that $x \notin \mathrm{Clo}\,(\mathcal{D} \setminus \{x\})$, which, by Def.1, shows that $x \notin \mathrm{Acc}\mathcal{D}$, contrary to the assumption. ■

Definition 2: *Let $\mathcal{D}, \mathcal{D}'$ be subsets of flat spaces $\mathcal{E}, \mathcal{E}'$ and let $x \in \mathrm{Acc}\mathcal{D}$. We say that $\varphi : \mathcal{D} \to \mathcal{D}'$ has the* (by Prop.1 unique) **limit** *$x' \in \mathcal{E}'$ at x if the mapping $\overline{\varphi}$ defined in Prop.1 is continuous at x'. We write*

$$x' = \lim_x \varphi = \lim_{z \to x} \varphi(z) \tag{57.2}$$

to express the assertion that φ has the limit x' at x.

If we apply Prop.1 to the case when $x \in \mathcal{D}$ we obtain

Proposition 2: *A given mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ is continuous at $x \in \mathcal{D}$ if and only if either x is an isolated point of $\mathcal{D}$, or else $x \in \mathrm{Acc}\mathcal{D}$ and $\lim_x \varphi = \varphi(x)$.*

The following characterization of the limit is an immediate consequence of Def.2.

Proposition 3: *The mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ has the limit x' at $x \in \mathrm{Acc}\mathcal{D}$ if and only if for every $\mathcal{N}' \in \mathrm{Nhd}_{x'}(\mathcal{E}')$, there is a $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$ such that*

$$\varphi_{>}(\mathcal{N} \cap (\mathcal{D} \setminus \{x\})) \subset \mathcal{N}'. \tag{57.3}$$

Using Prop.1 of Sect.56, we get still another characterization of the limit:

Proposition 4: *Let norms ν and ν' on the translation spaces $\mathcal{V}$ and $\mathcal{V}'$ of $\mathcal{E}$ and $\mathcal{E}'$ be given. The mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ has the limit x' at $x \in \mathrm{Acc}\mathcal{D}$ if and only if for every $\varepsilon \in \mathbb{P}^\times$ there is a $\delta \in \mathbb{P}^\times$ such that*

$$\big(0 < \nu(y - x) < \delta \implies \nu'(\varphi(y) - x') < \varepsilon\big) \quad \text{for all} \quad y \in \mathcal{D}. \tag{57.4}$$

Proposition 5: *If $\varphi : \mathcal{D} \to \mathcal{D}'$ and $x \in \mathrm{Acc}\mathcal{D}$ are such that φ has a limit at x, then*

$$\lim_x \varphi \in \mathrm{Clo}\,(\mathrm{Rng}\,\varphi) \tag{57.5}$$

and hence $\lim_x \varphi \in \mathcal{D}'$ or $\lim_x \varphi \in \mathrm{Acc}\mathcal{D}'$ (or both).

Proof: Put $x' := \lim_x \varphi$. If $\mathcal{N}' \in \mathrm{Nhd}_{x'}(\mathcal{E}')$ is given, then (57.3) holds for a suitable $\mathcal{N} \in \mathrm{Nhd}_x(\mathcal{E})$. Since $x \in \mathrm{Acc}\mathcal{D}$ we have $\mathcal{N} \cap (\mathcal{D} \setminus \{x\}) \neq \emptyset$ and hence, by (57.3), $\mathcal{N}' \cap \mathrm{Rng}\,\varphi \neq \emptyset$. Since $\mathcal{N}' \in \mathrm{Nhd}_x(\mathcal{E}')$ was arbitrary, (57.5) follows. ∎

The following result follows immediately from Def.2, the Composition Theorem for Continuity of Sect.56, and Prop.5.

Proposition 6: *Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be subsets of flat spaces $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ and let $\varphi : \mathcal{D} \to \mathcal{D}'$ and $\psi : \mathcal{D}' \to \mathcal{D}''$ be given. Assume, also, that φ has a limit at a given point $x \in \mathrm{Acc}\mathcal{D}$.*

If $\lim_x \varphi \in \mathcal{D}'$ and if ψ is continuous at $\lim_x \varphi$, then

$$\lim_x (\psi \circ \varphi) = \psi(\lim_x \varphi). \tag{57.6}$$

If $\lim_x \varphi \notin \mathcal{D}'$ then $\lim_x \varphi \in \mathrm{Acc}\mathcal{D}'$. If, in this case, ψ has a limit at $\lim_x \varphi$, then

$$\lim_x (\psi \circ \varphi) = \lim_{\lim_x \varphi} \psi. \tag{57.7}$$

Using Prop.2 of Sect.56, one immediately obtains the following result from Def.2 and Prop.1.

Proposition 7: *The mapping $\varphi : \mathcal{D} \to \mathcal{D}'$ has the limit x' at $x \in \mathrm{Acc}\mathcal{D}$ if and only if, for every sequence s in $\mathcal{D} \setminus \{x\}$ that converges to x, the sequence $\varphi \circ s$ converges to x', i.e.*

$$\lim\,(\varphi \circ s) = \lim_{(\lim\, s)} \varphi. \tag{57.8}$$

58 Compactness

Recall that a collection $\mathfrak{G}$ of subsets of a set $\mathcal{E}$ is said to *cover* a given set $\mathcal{K} \in \mathrm{Sub}\,\mathcal{E}$ if $\mathcal{K} \subset \bigcup \mathfrak{G}$ (see Sect.01).

Definition 1: *We say that a subset $\mathcal{K}$ of a flat space $\mathcal{E}$ is* **compact** *if every collection of open subsets of $\mathcal{E}$ that covers $\mathcal{K}$ has a finite subcollection that still covers $\mathcal{K}$.*

Proposition 1: *Let $\mathcal{D}$ be a subset of a flat space $\mathcal{E}$. A subset $\mathcal{K}$ of $\mathcal{D}$ is compact if and only if every collection that covers $\mathcal{K}$ and consists of sets open relative to $\mathcal{D}$ has a finite subcollection that still covers $\mathcal{K}$.*

Proof: Assume that $\mathcal{K} \subset \mathcal{D}$ is compact. Let $\mathfrak{H}$ be a collection that covers $\mathcal{K}$ and consists of sets open relative to $\mathcal{D}$. We then define the collection $\mathfrak{G}$ by $\mathfrak{G} := \{\mathcal{H}^\dagger \mid \mathcal{H} \in \mathfrak{H}\}$, where the open set $\mathcal{H}^\dagger$ is obtained from $\mathcal{H}$ by (56.2). Since $\mathcal{H} = \mathcal{D} \cap \mathcal{H}^\dagger \subset \mathcal{H}^\dagger$, it is clear that $\mathfrak{G}$ is a cover of $\mathcal{K}$ consisting of open sets. Since $\mathcal{K}$ is compact, we can determine a finite collection $\mathfrak{g} \subset \mathfrak{G}$ that covers $\mathcal{K}$. Then $\mathfrak{h} := \{\mathcal{D} \cap \mathcal{G} \mid \mathcal{G} \in \mathfrak{g}\}$ is a finite subcollection of $\mathfrak{H}$ that still covers $\mathcal{K}$.

Now assume that $\mathcal{K} \subset \mathcal{D}$ satisfies the covering condition. Let $\mathfrak{G}$ be a collection of open subsets of $\mathcal{E}$ that covers $\mathcal{K}$. Then $\mathfrak{H} := \{\mathcal{D} \cap \mathcal{G} \mid \mathcal{G} \in \mathfrak{G}\}$ is a collection that covers $\mathcal{K}$ and consists of sets open relative to $\mathcal{D}$. We determine a finite collection $\mathfrak{h} \subset \mathfrak{H}$ that still covers $\mathcal{K}$. For each $\mathcal{H} \in \mathfrak{h}$, we choose $\mathcal{G} \in \mathfrak{G}$ such that $\mathcal{H} = \mathcal{D} \cap \mathcal{G}$. The subcollection $\mathfrak{g}$ of all $\mathcal{G} \in \mathfrak{G}$ obtained in this way is finite and still covers $\mathcal{K}$. Since the collection $\mathfrak{G}$ was arbitrary, it follows that $\mathcal{K}$ is compact. ■

The importance of the concept of compactness becomes apparent in the following two theorems.

Compact Image Theorem: *The image of every compact set under a continuous mapping is again a compact set. More precisely, if $\mathcal{D}, \mathcal{D}'$ are subsets of flat spaces $\mathcal{E}, \mathcal{E}'$ and if $\varphi : \mathcal{D} \to \mathcal{D}'$ is continuous, then $\varphi_>(\mathcal{K})$ is compact for every compact subset $\mathcal{K}$ of $\mathcal{D}$.*

Proof: Assume that $\mathcal{K} \in \operatorname{Sub}\mathcal{D}$ is compact. Let $\mathfrak{H}'$ be a collection that covers $\varphi_>(\mathcal{K})$ and consists of sets open relative to $\mathcal{D}'$. By Prop.3 of Sect.56, the collection $\mathfrak{H} := \{\varphi^<(\mathcal{H}') \mid \mathcal{H}' \in \mathfrak{H}'\}$ consists of sets open relative to $\mathcal{D}$. Also, we have

$$\mathcal{K} \subset \varphi^<(\varphi_>(\mathcal{K})) \subset \varphi^<(\bigcup \mathfrak{H}') = \bigcup \mathfrak{H},$$

showing that $\mathfrak{H}$ covers $\mathcal{K}$. By Prop.1, since $\mathcal{K}$ is compact, we can determine a finite subcollection $\mathfrak{h}$ of $\mathfrak{H}$ that still covers $\mathcal{K}$. For each $\mathcal{H} \in \mathfrak{h}$ we can choose $\mathcal{H}' \in \mathfrak{H}'$ such that $\mathcal{H} = \varphi^<(\mathcal{H}')$. The subcollection $\mathfrak{h}'$ of $\mathfrak{H}'$ obtained in this way is finite and still covers $\varphi_>(\mathcal{K})$. Since the collection $\mathfrak{H}'$ was arbitrary, it follows from Prop.1 that $\varphi_>(\mathcal{K})$ is compact. ■

Uniform Continuity Theorem: *A continuous mapping with compact domain is necessarily uniformly continuous. More precisely, if $\mathcal{K}, \mathcal{D}'$ are subsets of flat spaces $\mathcal{E}, \mathcal{E}'$, if $\varphi : \mathcal{K} \to \mathcal{D}'$ is continuous, and if $\mathcal{K}$ is compact,* *then φ is uniformly continuous.*

Proof: We denote the translation spaces of $\mathcal{E}, \mathcal{E}'$ by $\mathcal{V}, \mathcal{V}'$.

Let $\mathcal{M}' \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V}')$ be given. We choose a norming cell $\mathcal{B}'$ in $\mathcal{V}'$ such that $\mathcal{B}' \subset \mathcal{M}'$. Then $(\varphi(x) + \frac{1}{2}\mathcal{B}') \cap \mathcal{D}' \in \mathrm{Nhd}_{\varphi(x)}(\mathcal{D}')$ for each $x \in \mathcal{K}$. Since φ is continuous we can choose, for each $x \in \mathcal{K}$, a norming cell $\mathcal{B}_x$ in $\mathcal{V}$ such that

$$\varphi_{>}((x + \mathcal{B}_x) \cap \mathcal{K}) \subset \varphi(x) + \frac{1}{2}\mathcal{B}'. \tag{58.1}$$

The collection $\{x + \frac{1}{2}\mathcal{B}_x \mid x \in \mathcal{K}\}$ consists of open sets and covers $\mathcal{K}$. Since $\mathcal{K}$ is compact, we may choose a finite subset $\mathfrak{k}$ of $\mathcal{K}$ such that $\{p + \frac{1}{2}\mathcal{B}_p \mid p \in \mathfrak{k}\}$ still covers $\mathcal{K}$. By Prop. 2 of Sect. 51, $\mathcal{B} := \bigcap\{\frac{1}{2}\mathcal{B}_p \mid p \in \mathfrak{k}\}$ is a norming cell in $\mathcal{V}$.

Now let $x, y \in \mathcal{K}$ be given. Since $\{p + \frac{1}{2}\mathcal{B}_p \mid p \in \mathfrak{k}\}$ covers $\mathcal{K}$, we may choose $p \in \mathcal{K}$ such that $x \in p + \frac{1}{2}\mathcal{B}_p \subset p + \mathcal{B}_p$. If $y - x \in \mathcal{B}$, we have

$$y = x + (y - x) \in p + \frac{1}{2}\mathcal{B}_p + \mathcal{B} \subset p + \frac{1}{2}\mathcal{B}_p + \frac{1}{2}\mathcal{B}_p = p + \mathcal{B}_p$$

and therefore $x, y \in (p + \mathcal{B}_p) \cap \mathcal{K}$. Hence, if we use (58.1) with x replaced by p, we obtain $\varphi(x), \varphi(y) \in \varphi(p) + \frac{1}{2}\mathcal{B}'$ and therefore

$$\varphi(y) - \varphi(x) \in (\varphi(p) + \frac{1}{2}\mathcal{B}') - (\varphi(p) + \frac{1}{2}\mathcal{B}') = \mathcal{B}'.$$

We have proved that, for all $x, y \in \mathcal{K}$, $y - x \in \mathcal{B}$ implies $\varphi(y) - \varphi(x) \in \mathcal{B}' \subset \mathcal{M}'$. Since $\mathcal{M}' \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V}')$ was arbitrary and since $\mathcal{B} \in \mathrm{Nhd}_{\mathbf{0}}(\mathcal{V})$ we conclude, according to Def.2 of Sect.56, that φ is uniformly continuous. ∎

The condition of Def.2 is usually very hard to verify for a given set $\mathcal{K}$. The following theorem gives an easy criterion.

Compactness Theorem: *A subset of a flat space is compact if and only if it is closed and bounded.*

Before proving this theorem we give some preliminary results, which are often useful in their own right.

Proposition 2: *Let $\mathcal{K}$ be a closed and bounded subset of $\mathcal{E}$ and let $\mathfrak{G}$ be a collection of open sets of $\mathcal{E}$ that covers $\mathcal{K}$. Given a norming cell $\mathcal{B}$ in the translation space $\mathcal{V}$ of $\mathcal{E}$, one can find a number $\sigma \in \mathbb{P}^{\times}$ with the following property: For every $x \in \mathcal{K}$ there is a $\mathcal{G} \in \mathfrak{G}$ such that $x + \sigma\mathcal{B} \subset \mathcal{G}$.*

Proof: Assume that the conclusion is false. We can then choose, for each $n \in \mathbb{N}^{\times}$, a point $p_n \in \mathcal{K}$ such that

$$p_n + \frac{1}{n}\mathcal{B} \not\subset \mathcal{G} \quad \text{for all} \quad \mathcal{G} \in \mathfrak{G}. \tag{58.2}$$

Since $\mathcal{K}$ and hence the sequence $p := (p_n \mid n \in \mathbb{N}^\times)$ in $\mathcal{K}$ is bounded, we can apply the Cluster Point Theorem and choose a cluster point of p, say z. Since $\mathcal{K}$ is closed, we can apply Prop.6 of Sect.55 to conclude that $z \in \mathcal{K}$. Since $\mathfrak{G}$ is a cover of $\mathcal{K}$, we may choose $\mathcal{G}_\circ \in \mathfrak{G}$ such that $z \in \mathcal{G}_\circ$. Since $\mathcal{G}_\circ$ is open and hence $\mathcal{G}_\circ \in \mathrm{Nhd}_z(\mathcal{E})$, we may choose $\varepsilon \in \mathbb{P}^\times$ such that $z + \varepsilon\mathcal{B} \subset \mathcal{G}_\circ$. Since z is a cluster point of p, we may choose $m \in \mathbb{N}^\times$ with $m \geq \frac{2}{\varepsilon}$ such that $p_m \in z + \frac{\varepsilon}{2}\mathcal{B}$. Since $(\frac{\varepsilon}{2} + \frac{1}{m}) \leq \varepsilon$, we obtain

$$p_m + \frac{1}{m}\mathcal{B} \subset z + \frac{\varepsilon}{2}\mathcal{B} + \frac{1}{m}\mathcal{B} \subset z + \varepsilon\mathcal{B} \subset \mathcal{G}_\circ,$$

which contradicts (58.2). ■

Proposition 3: *Let $\mathcal{K}$ be a subset of a flat space $\mathcal{E}$ with translation space $\mathcal{V}$. Then the following are equivalent:*

(i) *$\mathcal{K}$ is bounded.*

(ii) *For every norming cell $\mathcal{B}$ there is a finite subset $\mathfrak{k}$ of $\mathcal{K}$ such that $\mathcal{K} \subset \mathfrak{k} + \mathcal{B}$.*

(iii) *Every sequence in $\mathcal{K}$ has a cluster point.*

Proof: (i) $\Rightarrow$ (iii): This follows from the Cluster Point Theorem.

(iii) $\Rightarrow$ (ii): Assume that (ii) is false. We can then choose a norming cell $\mathcal{B}$ such that

$$\mathcal{K} \not\subset \mathfrak{k} + \mathcal{B} \quad \text{for every} \quad \mathfrak{k} \in \mathrm{Fin}\,\mathcal{K}. \tag{58.3}$$

We now define a sequence $p := (p_n \mid n \in \mathbb{N}^\times)$ in $\mathcal{K}$ by recursive choice as follows: Suppose $p|_{(n-1)^]}$ has been determined for a given $n \in \mathbb{N}^\times$. (Of course, when $n = 1$, since $0^] = \emptyset$, $p|_{(n-1)^]}$ is the empty list.) Since $\mathrm{Rng}\,(p|_{(n-1)^]})$ is a finite subset of $\mathcal{K}$, we have, by (58.3), $\mathcal{K} \not\subset \mathrm{Rng}\,(p|_{(n-1)^]}) + \mathcal{B}$. Hence we may choose $p_n \in \mathcal{K} \setminus (\mathrm{Rng}\,(p|_{(n-1)^]}) + \mathcal{B})$, which means that $p_n \in \mathcal{K}$ but $p_n \notin p_m + \mathcal{B}$ for all $m \in (n-1)^]$. It follows that $p_n - p_m \notin \mathcal{B}$ whenever $n, m \in \mathbb{N}^\times$ and $n > m$, which would not be possible if p had a cluster point. We conclude that (iii) is false.

(ii) $\Rightarrow$ (i): If (ii) holds then $\mathfrak{k} + \mathcal{B}$, being the union of the finite collection $\{p + \mathcal{B} \mid p \in \mathfrak{k}\}$ of bounded sets, is bounded. Hence $\mathcal{K}$, being a subset of $\mathfrak{k} + \mathcal{B}$ is also bounded. ■

Proof of Compactness Theorem: Assume that $\mathcal{K}$ is a compact subset of a given flat space $\mathcal{E}$ with translation space $\mathcal{V}$. Let $\mathcal{B}$ be a norming cell in $\mathcal{V}$. The collection $\{x + \mathcal{B} \mid x \in \mathcal{K}\}$ consists of open sets and covers $\mathcal{K}$. Hence we can choose a finite subset $\mathfrak{k}$ of $\mathcal{K}$ such that $\mathcal{K} \subset \bigcup\{p + \mathcal{B} \mid p \in \mathfrak{k}\} = \mathfrak{k} + \mathcal{B}$.

By (ii) $\Rightarrow$ (i) of Prop.3 it follows that $\mathcal{K}$ is bounded. To prove that $\mathcal{K}$ is closed, let $z \in \mathcal{E} \setminus \mathcal{K}$ be given. Then $\{\mathcal{E} \setminus (z + \rho\overline{\mathcal{B}} \mid \rho \in \mathbb{P}^\times\}$ is a collection of open sets whose union is $\mathcal{E} \setminus \{z\}$ and which, therefore, covers $\mathcal{K} \subset \mathcal{E} \setminus \{z\}$. Since $\mathcal{K}$ is compact, we may choose a non-empty finite subset F of $\mathbb{P}^\times$ such that

$$\mathcal{K} \subset \bigcup_{\rho \in F} \{\mathcal{E} \setminus (z + \rho\overline{\mathcal{B}})\} = \mathcal{E} \setminus (z + (\min F)\overline{\mathcal{B}}).$$

It follows that $(z + (\min F)\overline{\mathcal{B}}) \cap \mathcal{K} = \emptyset$. Since $z + (\min F)\overline{\mathcal{B}} \in \mathrm{Nhd}_z(\mathcal{E})$, we conclude that $z \notin \operatorname{Clo}\mathcal{K}$. Since $z \in \mathcal{E} \setminus \mathcal{K}$ was arbitrary, it follows that $\mathcal{E} \setminus \mathcal{K} \subset \mathcal{E} \setminus \operatorname{Clo}\mathcal{K}$ and hence that $\mathcal{K}$ is closed.

Assume that $\mathcal{K}$ is closed and bounded. Let a norming cell $\mathcal{B}$ and a collection $\mathfrak{G}$ of open subsets of $\mathcal{E}$ that covers $\mathcal{K}$ be given. We determine $\sigma \in \mathbb{P}^\times$ according to Prop.2. In view of Prop.3, (i) $\Rightarrow$ (ii), we may choose a finite subset $\mathfrak{k}$ of $\mathcal{K}$ such that $\mathcal{K} \subset \mathfrak{k} + \sigma\mathcal{B}$. By Prop.2, we may choose, for each $p \in \mathfrak{k}$, a set $\mathcal{G}_p \in \mathfrak{G}$ such that $p + \sigma\mathcal{B} \subset \mathcal{G}_p$. We then have

$$\mathcal{K} \subset \mathfrak{k} + \sigma\mathcal{B} = \bigcup_{p \in \mathfrak{k}} (p + \sigma\mathcal{B}) \subset \bigcup_{p \in \mathfrak{k}} \mathcal{G}_p,$$

which means that $\{\mathcal{G}_p \mid p \in \mathfrak{k}\}$ is a finite subcollection of $\mathfrak{G}$ that covers $\mathcal{K}$. Since $\mathfrak{G}$ was arbitrary, it follows that $\mathcal{K}$ is compact. ∎

The following is an immediate consequence of the Compactness Theorem and Prop.15 of Sect.53.

Proposition 4: *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are compact subsets of flat spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, then $\mathcal{K}_1 \times \mathcal{K}_2$ is a compact subset of $\mathcal{E}_1 \times \mathcal{E}_2$.*

Proposition 5: *If $\mathcal{K}$ is a compact subset of a flat space $\mathcal{E}$, and $\mathcal{B}$ a compact subset of the translation space $\mathcal{V}$ of $\mathcal{E}$, then $\mathcal{K} + \mathcal{B}$ is a compact subset of $\mathcal{E}$.*

Proof: We recall that the mapping $((x, \mathbf{v}) \mapsto x + \mathbf{v}) : \mathcal{E} \times \mathcal{V} \to \mathcal{E}$ is flat (Example 7 in Sect.33) and hence, by Prop.5 of Sect.56, continuous. Since $\mathcal{K} + \mathcal{B}$ is the image of $\mathcal{K} \times \mathcal{B}$ under this mapping, and since $\mathcal{K} \times \mathcal{B}$ is compact by Prop.4, the desired result follows from the Compact Image Theorem. ∎

Since every non-empty closed and bounded subset of $\mathbb{R}$ has a maximum and a minimum, we have the following direct consequence of the Compactness Theorem and the Compact Image Theorem.

Theorem on Attainment of Extrema: *A continuous real-valued function whose domain is non-empty, closed, and bounded, attains a maximum and a minimum. More precisely, if $\mathcal{K}$ is a non-empty closed and bounded subset of a flat space and if $f : \mathcal{K} \to \mathbb{R}$ is continuous, then there* *are points z and y in $\mathcal{K}$ such that*

$$f(z) \leq f(x) \leq f(y) \quad \text{for all} \quad x \in \mathcal{K}.$$

Proposition 6: *Let $\mathcal{D}$ be an open subset of a flat space $\mathcal{E}$ with translation space $\mathcal{V}$. For every compact subset $\mathcal{K}$ of $\mathcal{D}$ and every norm ν on $\mathcal{V}$, there is a $\delta \in \mathbb{P}^\times$ such that*

$$\mathcal{K} + \delta\overline{\mathrm{Ce}}(\nu) \subset \mathcal{D}. \tag{58.4}$$

The set $\mathcal{K} + \delta\overline{\mathrm{Ce}}(\nu)$ is compact.

Proof: Let a compact subset $\mathcal{K}$ of $\mathcal{D}$ and a norm ν on $\mathcal{V}$ be given. If $\mathcal{D} = \mathcal{E}$ then (58.4) is trivial. We assume now that $\mathcal{D} \neq \mathcal{E}$ and put $\mathcal{S} := \mathcal{E} \setminus \mathcal{D}$. We define $d : \mathcal{E} \to \mathbb{P}$ as in Prop.6 of Sect.56. Since $\mathcal{S}$ is closed by Prop.5 of Sect.53 it follows from Prop.6 of Sect.56 that $\mathcal{S} = d^<(\{0\})$ and hence $d_>(\mathcal{K}) \subset d_>(\mathcal{D}) \subset \mathbb{P}^\times$. Since $d|_\mathcal{K}$ is continuous, by the Theorem on Attainment of Extrema it attains a minimum $\sigma \in \mathbb{P}^\times$. We choose $\delta \in]0, \sigma[$. ($\delta := \frac{1}{2}\sigma$ would do.) Then $d_>(\mathcal{K}) \cap [\,0, \delta] = \emptyset$ and hence $\mathcal{K} \cap d^<([0, \delta]) = \emptyset$. By (56.11), it follows that $\mathcal{K} \cap (\mathcal{S} + \delta\overline{\mathrm{Ce}}(\nu)) = \emptyset$ and hence $\emptyset = (\mathcal{K} + \delta\overline{\mathrm{Ce}}(\nu)) \cap \mathcal{S} = (\mathcal{K} + \delta\overline{\mathrm{Ce}}(\nu)) \cap (\mathcal{E} \setminus \mathcal{D})$, which is equivalent to (58.4).

The Compactness of $\mathcal{K} + \delta\overline{\mathrm{Ce}}(\nu)$ is a consequence of Prop.5. ∎

We now assume that $\mathcal{D}$ and $\mathcal{D}'$ are subsets of flat spaces $\mathcal{E}$ and $\mathcal{E}'$ with translation spaces $\mathcal{V}$ and $\mathcal{V}'$, respectively. Recall Def.3 of Sect.55.

Proposition 7: *Let σ be a sequence in $\mathrm{Map}\,(\mathcal{D}, \mathcal{D}')$ that converges locally uniformly to φ. Then, for every compact subset $\mathcal{K}$ of $\mathcal{D}$, the sequence $(\sigma_n\,|_\mathcal{K} \mid n \in \mathbb{N}^\times)$ converges uniformly to $\varphi|_\mathcal{K}$.*

Proof: Let $\mathcal{K}$ be a non-empty compact subset of $\mathcal{D}$. For every $q \in \mathcal{K}$ we can choose an open neighborhood $\mathcal{G}_q$ of q such that the sequence $(\sigma_n|_{\mathcal{G}_q} \mid n \in \mathbb{N}^\times)$ converges uniformly. Then $\{\mathcal{G}_q \mid q \in \mathcal{K}\}$ is a collection of open sets that covers $\mathcal{K}$. Since $\mathcal{K}$ is compact, we can choose a finite subset $\mathfrak{k}$ of $\mathcal{K}$ such that $\{\mathcal{G}_q \mid q \in \mathfrak{k}\}$ still covers $\mathcal{K}$. Now let $\mathcal{M}' \in \mathrm{Nhd}_\mathbf{0}(\mathcal{V}')$ be given. For every $q \in \mathfrak{k}$ we can then determine $r_q \in \mathbb{N}^\times$ such that

$$\sigma_m(x) \in \varphi(x) + \mathcal{M}' \quad \text{for all} \quad m \in r_q + \mathbb{N},\ x \in \mathcal{G}_q \cap \mathcal{D}. \tag{58.5}$$

We note that $\mathfrak{k} \neq \emptyset$ and put $n := \max\{r_q \mid q \in \mathfrak{k}\}$. Since $m \in n + \mathbb{N}$ implies $m \in r_q + \mathbb{N}$ for all $q \in \mathfrak{k}$, it follows from (58.5) that $\sigma_m(x) \in \varphi(x) + \mathcal{M}'$ for all $m \in n + \mathcal{N}$ and all $x \in \mathcal{K}$. Since $\mathcal{M}' \in \mathrm{Nhd}_\mathbf{0}(\mathcal{V}')$ was arbitrary, it follows that $\{\sigma_n\,|_\mathcal{K} \mid n \in \mathbb{N}^\times\}$ converges uniformly. ∎

Notes 58

(1) What we call the "Compactness Theorem" is often called the "Heine-Borel Theorem", the "Borel-Lebesgue Theorem", or the "Heine-Borel-Lebesgue Theorem".

(2) Prop.2 is sometimes called "Lebesgue's Covering Lemma" and a number $\sigma \in \mathbb{P}^\times$ that has the property mentioned in Prop.2 is often called a "Lebesgue number" of a given covering collection $\mathfrak{G}$.

59 Problems for Chapter 5

(1) Let $\mathcal{S}$ be a bounded subset of a linear space $\mathcal{V}$ such that $\text{Lsp}\mathcal{S} = \mathcal{V}$. Show that

$$\mathcal{B} := \text{Int}\,\text{Cxh}(\mathcal{S} \cup (-\mathcal{S})) \tag{P5.1}$$

is a norming cell in $\mathcal{V}$.

(2) Show that the boundary of a box in a flat space $\mathcal{E}$ of dimension n is the union of a collection of $2n$ closures of boxes in hyperplanes in $\mathcal{E}$.

(3) Let $\mathcal{E}$ be a flat space and let $q \in \mathcal{E}$ be given. Show that a subset $\mathcal{S}$ of $\mathcal{E}$ is bounded if and only if, for every $\mathcal{N} \in \text{Nhd}_{\mathbf{0}}(\mathcal{V})$ one can find $\sigma \in \mathbb{P}^\times$ such that $\sigma(\mathcal{S} - q) \subset \mathcal{N}$.

(4) Let $\mathcal{S}$ be a non-empty subset of a flat space $\mathcal{E}$. Let ν be a norm on $\mathcal{V} := \mathcal{E} - \mathcal{E}$. Prove that, for every $\delta \in \mathbb{P}^\times$,

$$\text{diam}_\nu(\mathcal{S} + \delta\overline{\text{Ce}}(\nu)) = \text{diam}_\nu(\mathcal{S} + \delta\text{Ce}(\nu)) = \text{diam}_\nu(\mathcal{S}) + 2\delta, \tag{P5.2}$$

where $\text{Ce}(\nu)$ and $\overline{\text{Ce}}(\nu)$ are given by (51.8) and (51.9), and where the diameter diam_ν is defined by (52.1).

(5) Let $\mathcal{V}$ and $\mathcal{V}'$ be genuine inner-product spaces with $0 < \dim\mathcal{V} \leq \dim\mathcal{V}'$. Show that

$$||\mathbf{R}|| = 1 \quad \text{and} \quad |\mathbf{R}| = \sqrt{\dim\mathcal{V}}$$

for all $\mathbf{R} \in \text{Orth}(\mathcal{V}, \mathcal{V}')$.

(6) Let $\mathcal{V}$ be a linear space.

(a) Prove: If $\mathbf{f} := (\mathbf{f}_i \mid i \in I)$ is a finite family in $\mathcal{V}$ and

$$\mathcal{C} := \mathrm{Cxh}((\mathrm{Rng}\,\mathbf{f}) \cup (-\mathrm{Rng}\,\mathbf{f})), \tag{P5.3}$$

then, for all $\mathbf{v} \in \mathrm{Rng}\,\mathrm{lnc}_{\mathbf{f}}$,

$$\inf\{t \in \mathbb{P} \mid \mathbf{v} \in t\mathcal{C}\} = \inf\left\{\sum_{i\in I}|\lambda_i| \;\middle|\; \lambda \in \mathrm{lnc}_{\mathbf{f}}^{<}(\{\mathbf{v}\})\right\} \tag{P5.4}$$

(see Sect.37).

(b) Let $\boldsymbol{\beta} := (\boldsymbol{\beta}_i \mid i \in I)$ be a finite family in $(\mathcal{V}^*)^{\times}$ that spans $\mathcal{V}^*$. Show that $\nu : \mathcal{V} \to \mathbb{P}$, defined by

$$\nu(\mathbf{v}) := \max\{|\boldsymbol{\beta}_i\mathbf{v}| \mid i \in I\} \tag{P5.5}$$

is a norm on $\mathcal{V}$ and that the corresponding norming cell is given by

$$\mathrm{Ce}(\nu) = \bigcap_{i\in I}\boldsymbol{\beta}_i^{<}(\,]-1,1[\,). \tag{P5.6}$$

(c) Let the norm ν on $\mathcal{V}$ be defined as in Part (b). Show that its dual ν^* (see Sect.52) is given by

$$\nu^*(\boldsymbol{\lambda}) = \inf\left\{\sum_{i\in I}|\alpha_i| \;\middle|\; \alpha \in \mathrm{lnc}_{\boldsymbol{\beta}}{}^{<}(\{\boldsymbol{\lambda}\})\right\} \tag{P5.7}$$

for all $\boldsymbol{\lambda} \in \mathcal{V}^*$. Also, show that $\nu^* = \mathrm{no}_{\mathcal{B}^*}$ when

$$\mathcal{B}^* := \mathrm{Int}\,\mathrm{Cxh}((\mathrm{Rng}\,\boldsymbol{\beta}) \cup (-\mathrm{Rng}\,\boldsymbol{\beta})). \tag{P5.8}$$

(Hint: Use Part (a).)

(d) Using Part (c), show that (52.18) holds when $\mathbf{b}$ is a basis of $\mathcal{V}$ and that

$$\mathrm{Dmd}(\mathbf{b}) = \mathrm{Int}\,\mathrm{Cxh}((\mathrm{Rng}\,\mathbf{b}) \cup (-\mathrm{Rng}\,\mathbf{b})). \tag{P5.9}$$

(7) Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and let ν and ω be norms on $\mathcal{V}$ and $\mathcal{W}$, respectively.

(a) Show that

$$||\mathbf{w} \otimes \boldsymbol{\lambda}||_{\nu,\omega} = \omega(\mathbf{w})\nu^*(\boldsymbol{\lambda}) \tag{P5.10}$$

for all $\boldsymbol{\lambda} \in \mathcal{V}^*, \mathbf{w} \in \mathcal{W}$.

(b) Let $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{W})$ and bases $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ and $\mathbf{c} := (\mathbf{c}_j \mid j \in J)$ of $\mathcal{V}$ and $\mathcal{W}$, respectively, be given. Let $M \in \mathbb{R}^{J \times I}$ be the matrix of $\mathbf{L}$ relative to $\mathbf{b}$ and $\mathbf{c}$ (see Sect.16). Show that

$$||\mathbf{L}||_{\nu,\omega} \leq \alpha\beta \sum_{(j,i) \in J \times I} |M_{j,i}|, \tag{P5.11}$$

where

$$\alpha := \max\{\nu^*(\mathbf{b}_i^*) \mid i \in I\}, \;\; \beta := \max(\omega(\mathbf{c}_j) \mid j \in J). \tag{P5.12}$$

(8) Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and let $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{W})$ and bases $\mathbf{b}$ and $\mathbf{c}$ be given as in Part (b) of Problem 7.

(a) Prove that

$$\|\mathbf{L}\|_{\nu,\omega} = \max\left\{ \sum_{j \in J} |M_{j,i}| \;\middle|\; i \in I \right\} \tag{P5.13}$$

when $\nu := \mathrm{no}_{\mathrm{Dmd}(\mathbf{b})}, \;\; \omega := \mathrm{no}_{\mathrm{Dmd}(\mathbf{c})}$.

(b) Prove that

$$||\mathbf{L}||_{\nu,\omega} = \max\left\{ \sum_{i \in I} |M_{j,i}| \;\middle|\; j \in J \right\} \tag{P5.14}$$

when $\nu := \mathrm{no}_{\mathrm{Box}(\mathbf{b})}$, $\omega := \mathrm{no}_{\mathrm{Box}(\mathbf{c})}$. (Hint: Use Part (a) and Prop.7 of Sect.52.)

(9) Let $\mathcal{E}$ be a flat space and let (x, y, z) be a flatly independent triple of points in $\mathcal{E}$. Also, assume that ρ, σ and τ are sequences in $\mathbb{R}$ and indexed on $\mathbb{N}^\times$ such that

$$\rho_n + \sigma_n + \tau_n = 1 \quad \text{for all} \quad n \in \mathbb{N}^\times, \tag{P5.15}$$

so that

$$s_n := \rho_n x + \sigma_n y + \tau_n z \tag{P5.16}$$

is meaningful for each $n \in \mathbb{N}^\times$ as a symbolic sum (see (35.5)).

(a) Show that the sequence $s := (s_n \,|\, n \in \mathbb{N}^\times)$ converges if and only if both ρ and σ converge.

(b) Show: If s has a cluster point, so do ρ, σ and τ.

(c) Give an example of sequences ρ, σ, τ satisfying $(P5.15)$ such that ρ, σ, τ all have cluster points but the sequence s, as defined by $(P5.16)$, has no cluster point.

(10) Let a flat space $\mathcal{E}$, a point $q \in \mathcal{E}$, and a norm ν on $\mathcal{V} := \mathcal{E} - \mathcal{E}$ be given. Consider the ν-cell $\mathcal{C} := q + \mathrm{Ce}(\nu)$ with closure $\overline{\mathcal{C}} := q + \overline{\mathrm{Ce}}(\nu)$ (see Def.4 of Sect.51). For each $n \in \mathbb{N}$ define $f_n : \overline{\mathcal{C}} \to \mathbb{R}$ by

$$f_n(x) = (\nu(x-q))^n \quad \text{for all} \quad x \in \overline{\mathcal{C}}. \tag{P5.17}$$

(a) Show that the sequence $f := (f_n \,|\, n \in \mathbb{N})$ in $\mathrm{Map}\,(\overline{\mathcal{C}}, \mathbb{R})$ converges and find its limit.

(b) Show that the sequence f does not converge locally uniformly.

(c) Show that the sequence $(f_n|_{\mathcal{C}} \mid n \in \mathbb{N})$ converges locally uniformly but not uniformly.

(11) Let $\mathcal{S}$ be a subset of a flat space $\mathcal{E}$ and let $\mathcal{C}$ be a closed subset of $\mathcal{V} := \mathcal{E} - \mathcal{E}$.

(a) Show that $\mathcal{S} + \mathcal{C}$ is closed if $\mathcal{S}$ is compact.

(b) Give a counterexample which shows that $\mathcal{S} + \mathcal{C}$ need not be closed if $\mathcal{S}$ is closed.

(12) Let $\mathcal{K}$ be a compact subset of a flat space $\mathcal{E}$.

(a) Let $(\mathcal{G}_n \mid n \in \mathbb{N}^\times)$ be a sequence in $\mathrm{Sub}\,\mathcal{K}$ such that $\mathcal{G}_n$ is open relative to $\mathcal{K}$ for each $n \in \mathbb{N}^\times$,

$$\mathcal{G}_{n+1} \supset \mathcal{G}_n \quad \text{for all} \quad n \in \mathbb{N}^\times,$$

and $\bigcup(\mathcal{G}_n \mid n \in \mathbb{N}^\times) = \mathcal{K}$. Show that there is $m \in \mathbb{N}^\times$ such that $\mathcal{K} = \mathcal{G}_m$.

(b) Let $f := (f_n \mid n \in \mathbb{N}^\times)$ be a sequence in $\mathrm{Map}\,(\mathcal{K}, \mathbb{R})$ such that f_n is continuous for each $n \in \mathbb{N}^\times$,

$$f_{n+1} \leq f_n \quad \text{(value-wise) for all } n \in \mathbb{N}^\times,$$

and f converges (value-wise) to a continuous function $g \in \operatorname{Map}(\mathcal{K}, \mathbb{R})$. Prove that f converges uniformly to g. (Hint: Apply Part (a) to the case when

$$\mathcal{G}_n := (f_n - g)^<([0, \varepsilon[\,), \quad \text{where} \quad \varepsilon \in \mathbb{P}^\times.)$$

Note: The result of Part (b) is usually called "Dini's Theorem". ■

(13) Let $\mathcal{E}$ and $\mathcal{E}'$ be flat spaces and let $\varphi : \mathcal{E} \to \mathcal{E}'$ be a continuous mapping. Show: If the pre-image under φ of every bounded subset of $\mathcal{E}'$ is bounded, then $\operatorname{Rng} \varphi$ is a closed subset of $\mathcal{E}'$. (Hint: Use Prop.6 of Sect.55, the Cluster Point Theorem, and Prop.2 of Sect.56.)